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10. Tutorial Analysis II for MCS
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(T10.1) Solution.

(i) Recall from (T12.1) from Analysis I that we say that A is totally bounded if for all $\varepsilon > 0$ there exists a finite subset $\{a_1, \dots, a_n\} \subseteq A$ s.t.

$$A = \bigcup_{j=1}^{n} \{x \in A : \|x - a_j\|_\infty < \varepsilon\},$$

i.e. such that

$$A \subseteq \bigcup_{j=1}^{n} \{x \in K^n : \|x - a_j\|_\infty < \varepsilon\}.$$

If A is totally bounded we let $N \in \mathbb{N}$ and $\{a_1, \dots, a_N\} \subseteq A$ be s.t.

$$A = \bigcup_{j=1}^{N} \{x \in A : \|x - a_j\|_\infty < 1\}.$$

Let $M := \max\{\|a_i - a_j\|_\infty : 1 \leq i, j \leq N\} + 2$.

Then by the triangle inequality we have $\|x - y\|_\infty < M$ for all $x, y \in A$, so A is bounded.

Let now A be bounded, and let $M \in \mathbb{R}$ be s.t. $\|x-y\|_\infty < M$ for all $x, y \in A$. By (T12.1) in Analysis I we have that A is totally bounded if and only if every sequence in A has a Cauchy subsequence. Recall that a nonempty bounded subset of $\mathbb{R}$ is totally bounded. We first show that if $\emptyset \neq B \subseteq \mathbb{C}$ is bounded then B is totally bounded. Let $(z_n)_{n \in \mathbb{N}}$ be a sequence in B with $z_n = x_n + i y_n$ and with x_n and y_n real numbers. (For $n \in \mathbb{N}$.) Then $(x_n)_n$ is a sequence in a bounded subset of $\mathbb{R}$. So let $(x_{n_k})_{k \in \mathbb{N}}$ be a Cauchy subsequence of $(x_n)_n$. Then $(y_{n_k})_{k \in \mathbb{N}}$ is a sequence in a bounded subset of $\mathbb{R}$. So let $(y_{n_{k_m}})_{m \in \mathbb{N}}$ be a Cauchy subsequence of $(y_{n_k})_k$. Then also $(x_{n_{k_m}})_{m \in \mathbb{N}}$ is a Cauchy sequence, so $(z_{n_{k_m}})_{m \in \mathbb{N}}$ is a Cauchy subsequence of $(z_n)_{n \in \mathbb{N}}$. Thus B is totally bounded.

We want to show by induction on n that

$$(*) \qquad \emptyset \neq A \subseteq \mathbb{K}^n \text{ bounded} \Rightarrow \emptyset \neq A \subseteq \mathbb{K}^n \text{ totally bounded}$$

holds for all $n \in \mathbb{N}$. The case $n = 1$ is dealt with above. Suppose we have proved that $(*)$ holds for n.

Let $(x_m)_{m\in\mathbb{N}}$ be a sequence in $A \subseteq \mathbb{K}^{n+1}$ with $x_m = x_m^1 e_1 + \dots + x_m^{n+1} e_{n+1}$, where $\{e_1, \dots, e_{n+1}\}$ is the standard basis of $\mathbb{K}^{n+1}$ and $x_m^1, \dots, x_m^{n+1} \in \mathbb{K}$. (The superscripts are for notational convenience, and do not signify powers of x_m.) Now $(x_m^{n+1})_{m\in\mathbb{N}}$ is a sequence in a bounded subset of $\mathbb{K}$. For suppose we could find $i, j \in \mathbb{N}$ s.t. $|x_i^{n+1} - x_j^{n+1}| \geq M$. Then

$$\|x_i - x_j\|_\infty = \max\left\{|x_i^1 - x_j^1|, \dots, |x_i^{n+1} - x_j^{n+1}|\right\} \geq M,$$

which contradicts the assumption. So $(x_m^{n+1})_{m\in\mathbb{N}}$ has a Cauchy subsequence $(x_{m_k}^{n+1})_{k\in\mathbb{N}}$. Then $(x_{m_k}^1 e_1 + \dots + x_{m_k}^n e_n)_{k\in\mathbb{N}}$ is a sequence in a bounded subset of $\mathbb{K}^n$ (properly speaking with the e_i now denoting the elements of the standard basis of $\mathbb{K}^n$). For suppose we could find $i, j \in \mathbb{N}$ s.t.

$$\|(x^1_{m_i}e_1+\ldots+x^n_{m_i}e_n)-(x^1_{m_j}e_1+\ldots+x^n_{m_j}e_n)\|_\infty \geq M.$$

Then

$$M \leq \|(x^1_{m_i}e_1+\ldots+x^n_{m_i}e_n)-(x^1_{m_j}e_1+\ldots+x^n_{m_j}e_n)\|_\infty$$

$$= \max\{|x^1_{m_i}-x^1_{m_j}|,\ldots,|x^n_{m_i}-x^n_{m_j}|\}$$

$$\leq \max\{|x^1_{m_i}-x^1_{m_j}|,\ldots,|x^n_{m_i}-x^n_{m_j}|,|x^{n+1}_{m_i}-x^{n+1}_{m_j}|\}$$

$$= \|x_{m_i}-x_{m_j}\|_\infty,$$

contradicting the assumption. Thus we can let $(x^1_{m_{k_i}}e_1+x^2_{m_{k_i}}e_2+\ldots+x^n_{m_{k_i}}e_n)_{i\in\mathbb{N}}$ be a Cauchy subsequence of $(x^1_{m_k}e_1+x^2_{m_k}e_2+\ldots+x^n_{m_k}e_n)_{k\in\mathbb{N}}$.

Then $(x^1_{m_{k_i}}e_1+\ldots+x^n_{m_{k_i}}e_n+x^{n+1}_{m_{k_i}}e_{n+1})_{i\in\mathbb{N}}$ is a Cauchy subsequence of $(x_m)_{m\in\mathbb{N}}$, since

$$\|x_i-x_j\|_\infty = \max\{|x^1_i-x^1_j|,\ldots,|x^{n+1}_i-x^{n+1}_j|\}$$

$$= \max\{\|(x^1_ie_1+\ldots+x^n_ie_n)-(x^1_je_1+\ldots+x^n_je_n)\|_\infty,|x^{n+1}_i-x^{n+1}_j|\}.$$

(And for a given $\varepsilon>0$ we can find $N\in\mathbb{N}$ such that $i,j\geq N$ gives $\|(x^1_{m_{k_i}}e_1+\ldots+x^n_{m_{k_i}}e_n)-(x^1_{m_{k_j}}e_1+\ldots+x^n_{m_{k_j}}e_n)\|_\infty<\varepsilon$ and $|x^{n+1}_{m_{k_i}}-x^{n+1}_{m_{k_j}}|<\varepsilon$.) Thus A is totally bounded.

(ii) Since $\mathbb{K}^n$ is complete we have from (T9.1) that

$$A \text{ closed in } \mathbb{K}^n \Leftrightarrow A \text{ complete}.$$

From (i) we have

$$A \text{ bounded} \Leftrightarrow A \text{ totally bounded}.$$

Thus

$$A \text{ closed and bounded}$$

$$\Leftrightarrow$$

$$A \text{ complete and totally bounded}$$

$$\overset{((\#14.1) \text{ in Analysis I})}{\Leftrightarrow}$$

$$A \text{ compact}.$$

$\square$

(T10.2) Solution.

By Theorem 6.42 it is no restriction to assume $V = \mathbb{K}^n$. (Since if $S : V \to \mathbb{K}^n$ is a bijective, linear and continuous transformation such that also S^{-1} is linear and continuous, then any linear transformation $T : V \to W$ gives that $T \circ S^{-1} : \mathbb{K}^n \to W$ is linear. If we then have proved the statement for linear transformations from $\mathbb{K}^n$ to W, then we can conclude that $T \circ S^{-1}$ is continuous. But then also $T = (T \circ S^{-1}) \circ S$ is continuous.)

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{K}^n$, and let $x, y \in \mathbb{K}^n$ with

$$x = x_1 e_1 + \ldots + x_n e_n,$$

$$y = y_1 e_1 + \ldots + y_n e_n,$$

$x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{K}$. Let $\varepsilon > 0$.

Then

$$\|T(x) - T(y)\| =$$

$$\|T(x_1 e_1 - y_1 e_1) + \ldots + T(x_n e_n - y_n e_n)\| \le$$

$$|x_1 - y_1| \|T(e_1)\| + \ldots + |x_n - y_n| \|T(e_n)\|$$

$$\le \|x - y\|_\infty (\|T(e_1)\| + \ldots + \|T(e_n)\|).$$

So let $x, y \in \mathbb{K}^n$ be s.t.

$$\|x - y\|_\infty < \frac{\varepsilon}{\|T(e_1)\| + \ldots + \|T(e_n)\|}.$$

Then $\|T(x) - T(y)\| < \varepsilon$. By Theorem 6.44 the choice of norm on $\mathbb{K}^n$ was immaterial. Thus T is continuous. $\square$