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9. Tutorial Analysis II for MCS
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(T9.1) Solution. (For a simpler proof using E3.4 (iv), see p. 18.)

(i) $\Rightarrow$ (ii): Assume that X is closed in Y.

That is, if $x \in Y$ is an accumulation point of X then $x \in X$. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X, i.e. $x_n \in X$ for all $n \in \mathbb{N}$ and

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n,m \geq N)(d(x_n, x_m) < \varepsilon).$$

Since Y is complete we know that $(x_n)_{n\in\mathbb{N}}$ converges in Y. Let $x \in Y$ be such that $\lim_{n\to\infty} x_n = x$. Assume $x \notin X$.

For each $\varepsilon > 0$ we get $n \in \mathbb{N}$ s.t. $d(x_n, x) < \varepsilon$. So each ε-neighborhood $U_\varepsilon(x)$ of x contains a point $x_n \in X$. Since $x \notin X$ and $x_n \in X$

we also have $x_n \neq x$, and x is therefore an accumulation point of X. But then $x \in X$, since X is closed in Y. This contradicts our assumption, and thus $x \in X$. So $(x_n)_{n \in \mathbb{N}}$ is convergent in X. Hence X is complete.

(ii) $\Rightarrow$ (i):

We assume that X is complete. We must show that if $x \in Y$ is an accumulation point of X then $x \in X$. Let $x \in Y$ be an accumulation point of X. Then

$$(\forall n \in \mathbb{N})\left(\left(\left(U_{\frac{1}{n}}(x) \setminus \{x\}\right) \cap X\right) \neq \emptyset\right),$$

so define $(x_n)_{n \in \mathbb{N}}$ by for $n \in \mathbb{N}$ letting

$$x_n \in \left(U_{\frac{1}{n}}(x) \setminus \{x\}\right) \cap X.$$

Then $(x_n)_{n \in \mathbb{N}}$ is Cauchy, and since X is complete we find $y \in X$ s.t.

$\lim_{n \to \infty} x_n = y$. Then $x = y$, so $x \in X$.

(Suppose $x \neq y$. Then $d(x,y) = \varepsilon > 0$.

Since $\lim_{n\to\infty} x_n = x$ in Y we find $n \in \mathbb{N}$ s.t. $m \geq n$ gives $d(x_m, x) < \frac{\varepsilon}{2}$.

Likewise we find $n' \in \mathbb{N}$ s.t. $m \geq n'$ gives $d(x_m, y) < \frac{\varepsilon}{2}$. Let $n_1 = \max\{n, n'\}$.

Then $d(x,y) \leq d(x, x_{n_1}) + d(x_{n_1}, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

This contradicts $d(x,y) = \varepsilon$, and so $x = y$.

Hence $x \in X$, and so X is closed in Y.) $\square$

(T9.2) Solution.

(i) We check the axioms for vector spaces:

(1) $(\forall S, T \in L(V,W))(S + T = T + S)$. Let $S, T \in L(V,W)$, $x \in V$. Then

$$(S+T)(x) = S(x) + T(x) = T(x) + S(x) = (T+S)(x).$$

So

$$(\forall S, T \in L(V,W))(S+T = T+S),$$ i.e. addition is commutative.

(2) Let $S, S_1, T \in L(V,W)$, $x \in V$. Then

$$((S+S_1)+T)(x) = (S+S_1)(x) + T(x)$$

$$= S(x) + S_1(x) + T(x)$$

$$= S(x) + (S_1+T)(x)$$

$$= (S+(S_1+T))(x),$$

and so

$$((S+S_1)+T) = (S+(S_1+T)).$$

So addition on $L(V,W)$ is associative.

(3) Let $0 : V \to W$ be the constant zero function, i.e. $0(x) = 0$ for all $x \in V$. Then $0 \in L(V,W)$ and

$(0+T)(x) = 0(x) + T(x) = T(x)$ for all $x \in V$.

So 0 is the neutral element with respect to addition on $L(V,W)$.

(4) Let T. Define $S \in L(V,W)$ by

$S(x) = -1 \cdot T(x)$ for all $x \in V$.

Then $S + T = 0$ and S

is the additive inverse of T in $L(V,W)$,

since
$$
\begin{aligned}
(S+T)(x) &= S(x) + T(x) \\
&= (-1 \cdot T(x)) + T(x) \\
&= 0 \\
&= 0(x).
\end{aligned}
$$

S is the composition of T and the continuous and linear function $x \mapsto -1 \cdot x : W \to W$,

so S is indeed an element of $L(V,W)$.

(5) $1 \in \mathbb{K}$ is the identity for scalar multiplication, for if we let $T \in L(V, W)$ and $x \in V$, then

$$(1 \cdot T)(x) = 1 \cdot T(x) = T(x),$$

since $1 \in \mathbb{K}$ is the identity for scalar multiplication in W.

That is,

$$1 \cdot T = T.$$

(6) Let $r, s \in \mathbb{K}$, $T \in L(V, W)$, $x \in V$.

Then

$$(r \cdot (s \cdot T))(x) = r \cdot (s \cdot T)(x)$$
$$= rs\, T(x)$$
$$= ((rs) \cdot T)(x),$$

and thus

$$(r \cdot (s \cdot T)) = ((rs) \cdot T).$$

So scalar multiplication is associative.

(7) Let $r \in \mathbb{K}$, $S, T \in L(V, W)$, $x \in V$.

Then

$$(r \cdot (S+T))(x) = r \cdot (S+T)(x)$$

$$= r \cdot (S(x) + T(x))$$

$$= r \cdot S(x) + r \cdot T(x)$$

$$= (r \cdot S)(x) + (r \cdot T)(x)$$

$$= (r \cdot S + r \cdot T)(x)$$

Hence $r \cdot (S+T) = r \cdot S + r \cdot T.$

(8) Let $r, s \in \mathbb{K}$, $T \in L(V, W)$, $x \in V$.

Then

$$((r+s) \cdot T)(x) = (r+s) T(x)$$

$$= r \cdot T(x) + s \cdot T(x)$$

$$= (r \cdot T)(x) + (s \cdot T)(x)$$

$$= ((r \cdot T) + (s \cdot T))(x),$$

so

$$(r+s) \cdot T = (r \cdot T) + (s \cdot T).$$

Thus $L(V,W)$ is a vector space over $\mathbb{K}$.

The solution to this exercise would be a lot less tedious if we had assumed it known that the set W^V of functions $f: V \to W$ is a vector space over $\mathbb{K}$. Then we would simply show that $L(V,W) \subseteq W^V$ is closed under addition and scalar multiplication.

(That this would be enough is a result from linear algebra.)

(ii) We first prove that, for any $T \in L(V,W)$, $\sup\{\|T(x)\| : x \in V, \|x\| \leq 1\}$ exists. By Theorem 6.23 in the handouts we know that there exists $M > 0$ s.t. $(\forall x \in V)\ (\|T(x)\| \leq M\|x\|)$, since T is continuous. Thus

$$(\forall x \in V)(\|x\| \leq 1 \Rightarrow \|T(x)\| \leq M), \quad \text{so}$$

$$\sup\{\|T(x)\| : x \in V, \|x\| \leq 1\} \leq M.$$

Next we prove that N1, N2, N3 from the definition of a norm are satisfied.

Since $\|T(x)\| \geq 0$ for any $T \in L(V,W)$ and any $x \in V$, we get $\|T\| \geq 0$ for any $T \in L(V,W)$. If $T = 0$ then

$\|T(x)\| = \|0(x)\| = \|0\| = 0$ for any $x \in V$, so $\|T\| = 0$. If $\|T\| = 0$ then $\|T(x)\| = 0$ for all $x \in V$ with $\|x\| \leq 1$. Let $y \in V$, $\|y\| \neq 0$. Then

$\left\| \frac{y}{\|y\|} \right\| = 1$, so $\left\| T\left(\frac{y}{\|y\|}\right) \right\| = 0$.

We have

$$\left\| T\left(\frac{y}{\|y\|}\right) \right\| = \left\| \frac{1}{\|y\|} \cdot T(y) \right\| = \left| \frac{1}{\|y\|} \right| \cdot \|T(y)\|,$$

so $\|T(y)\| = 0$ for any $y \in V$.

Thus N1 holds.

Let $\lambda \in \mathbb{K}$ and let $T \in L(V, W)$. Then

$$\|\lambda \cdot T\| = \sup\{\|(\lambda \cdot T)(x)\| : x \in V, \|x\| \leq 1\}$$

$$= \sup\{|\lambda| \cdot \|T(x)\| : x \in V, \|x\| \leq 1\}$$

$$= |\lambda| \cdot \sup\{\|T(x)\| : x \in V, \|x\| \leq 1\}$$

$$= |\lambda| \cdot \|T\|,$$

since $\|(\lambda \cdot T)(x)\| = \|\lambda \cdot T(x)\| = |\lambda| \, \|T(x)\|$.

Thus N2 holds.

Let now $S, T \in L(V, W)$. Then

$$\|S + T\| = \sup\{\|(S+T)(x)\| : x \in V, \|x\| \leq 1\}$$

$$= \sup\{\|S(x) + T(x)\| : x \in V, \|x\| \leq 1\}$$

$$\leq \sup\{\|S(x)\| + \|T(x)\| : x \in V, \|x\| \leq 1\}$$

$$\leq \sup\{\|S(x)\| : x \in V, \|x\| \leq 1\} + \sup\{\|T(x)\| : x \in V, \|x\| \leq 1\} = \|S\| + \|T\|.$$

So N3 holds, and $\|\cdot\|$ is a norm on $L(V,W)$.

Finally, we prove $\|T(x)\| \le \|T\|\cdot\|x\|$ for all $x \in V$. If $x=0$ this obivously holds, so assume $x \neq 0$, i.e. $\|x\| \neq 0$.

Then $\left\| \frac{x}{\|x\|} \right\| = 1$, so

$$\|T(x)\| = \left\| \|x\| \cdot T\left(\frac{x}{\|x\|}\right) \right\| = \|x\| \cdot \left\| T\left(\frac{x}{\|x\|}\right) \right\|$$

$$\le \|T\| \cdot \|x\|.$$

(iii) Suppose $(T_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L(V,W)$. We must show that $(T_n)_{n\in\mathbb{N}}$ converges. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ s.t.

$$(\forall m,n \ge N)\ (\|T_m - T_n\| < \varepsilon).$$

Then from (ii) it follows that for $x \in V$,

$$\|T_m(x) - T_n(x)\| \leq \|T_m - T_n\| \cdot \|x\| < \varepsilon \|x\|$$

for $m, n \geq N$, so

$(T_n(x))_{n \in \mathbb{N}}$ is a Cauchy sequence in W. Since W is a Banach space we get that $(T_n(x))_{n \in \mathbb{N}}$ converges in W. Let $\lim_{n \to \infty} T_n(x) = T(x)$.

This defines a function $T: V \to W$.

Now we prove $T \in L(V, W)$.

Let $x, y \in V$. Then

$T_n(x+y) = T_n(x) + T_n(y)$ for each $n \in \mathbb{N}$. Let $\varepsilon_1 > 0$. We can find $n_1 \in \mathbb{N}$ s.t. $n \geq n_1$ gives

$$\|T_n(x) - T(x)\| < \frac{\varepsilon_1}{3}.$$

Likewise we find $n_2 \in \mathbb{N}$ s.t. $n \geq n_2$ gives

$$\|T_n(y) - T(y)\| < \frac{\varepsilon_1}{3},$$

and also $n_3 \in \mathbb{N}$ s.t. $n \geq n_3$ gives

$$\|T_n(x+y) - T(x+y)\| < \frac{\varepsilon_1}{3}.$$

Let $N_1 = \max\{n_1, n_2, n_3\}$. Then for $n \geq N_1$

$$\|T(x+y) - (T(x) + T(y))\| =$$

$$\|T(x+y) - T_n(x+y) + (T_n(x) + T_n(y)) - (T(x) + T(y))\| \leq$$

$$\|T(x+y) - T_n(x+y)\| + \|T_n(x) - T(x)\| + \|T_n(y) - T(y)\| <$$

$$\frac{\varepsilon_1}{3} + \frac{\varepsilon_1}{3} + \frac{\varepsilon_1}{3} = \varepsilon_1.$$

Since ε_1 was arbitrary we get

(*) $$T(x+y) = T(x) + T(y).$$

Let now $x \in V$, $\lambda \in \mathbb{K}$. Then for each $n \in \mathbb{N}$

$$T_n(\lambda x) = \lambda \cdot T_n(x).$$

Let $\varepsilon_1 > 0$, and let $n_1 \in \mathbb{N}$ be such that $\|T_n(\lambda x) - T(\lambda x)\| < \frac{\varepsilon_1}{2}$ for $n \geq n_1$.

Let $n_2 \in \mathbb{N}$ be such that

$$\|\lambda T_n(x) - \lambda T(x)\| < \frac{\varepsilon_1}{2} \text{ for } n \geq n_2.$$

Let $N_1 = \max\{n_1, n_2\}$. Then for $n \geq N_1$

$$\|\lambda T(x) - T(\lambda x)\| =$$

$$\|\lambda T(x) - \lambda T_n(x) + T_n(\lambda x) - T(\lambda x)\| \leq$$

$$\|\lambda T(x) - \lambda T_n(x)\| + \|T_n(\lambda x) - T(\lambda x)\| <$$

$$\frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} = \varepsilon_1.$$

Since ε_1 was arbitrary we get

$$T(\lambda x) = \lambda T(x).$$

This and (*) together imply that T is a linear transformation.

We must next prove that T is continuous.

We saw that for $m, n \geq N$ we have

$$\|T_m(x) - T_n(x)\| < \varepsilon \|x\|$$

for all $x \in V$, so in particular

$$\|T_m(x) - T_n(x)\| < \varepsilon \text{ for } x \in V \text{ s.t. } \|x\| \leq 1.$$

Fix $n \geq N$, and suppose

$\|T(x) - T_n(x)\| > 2\varepsilon$ for some $x \in V$ with $\|x\| \leq 1$. Let $m \geq N$ be such that we for this $x \in V$ have

$\|T(x) - T_m(x)\| < \varepsilon$. Then

$$\|T(x) - T_n(x)\| = \|(T(x) - T_m(x)) + (T_m(x) - T_n(x))\| \leq$$

$$\|T(x) - T_m(x)\| + \|T_m(x) - T_n(x)\| < 2\varepsilon,$$

a contradiction. Thus $\|T(x) - T_n(x)\| \leq 2\varepsilon$ for all $x \in V$ with $\|x\| \leq 1$, for any $n \geq N$. It follows that in particular

$$\|T(x) - T_N(x)\| \leq 2\varepsilon$$

for any $x \in V$ with $\|x\| \leq 1$. We let $T - T_N : V \to W$ be defined by

$$(T - T_N)(x) = T(x) - T_N(x) \text{ for all } x \in V.$$

We have (for $x \in V$ with $\|x\| \neq 0$)

$$\|(T - T_N)(x)\| = \|T(x) - T_N(x)\|$$

$$= \left\| \|x\| \cdot T\left(\frac{x}{\|x\|}\right) - \|x\| \cdot T_N\left(\frac{x}{\|x\|}\right) \right\|$$

$$= \left\| \|x\| \left(T\left(\frac{x}{\|x\|}\right) - T_N\left(\frac{x}{\|x\|}\right) \right) \right\|$$

$$= \|x\| \cdot \left\| T\left(\frac{x}{\|x\|}\right) - T_N\left(\frac{x}{\|x\|}\right) \right\|$$

$$\leq \|x\| \, 2\varepsilon,$$

and thus $T - T_N$ is continuous by Theorem 6.23 in the handouts. (That $T - T_N$ is linear is trivial.)

Thus $T = (T - T_N) + T_N$ is also continuous, and so $T \in L(V, W)$.

Finally we must show that $(T_n)_{n \in \mathbb{N}}$ converges to T, i.e. that $\lim_{n \to \infty} \|T_n - T\| = 0$.

We have seen that $\|T(x) - T_n(x)\| \leq 2\varepsilon$ for all $n \geq N$ and for all $x \in V$ with $\|x\| \leq 1$.

Then

$$\|T_n - T\| = \sup\{\|(T_n - T)(x)\| : x \in V, \|x\| \leq 1\}$$

$$= \sup\{\|T_n(x) - T(x)\| : x \in V, \|x\| \leq 1\} \leq 2\varepsilon,$$

and so $\lim_{n \to \infty} \|T_n - T\| = 0$.

So $L(V, W)$ is a Banach space.

$\square$

(T9.1) Solution. (Alternative.)

(i) $\Rightarrow$ (ii)

Assume that X is closed in Y. Then if $(x_n)_{n\in\mathbb{N}}$ is a sequence with $x_n \in X$ and $\lim_{n\to\infty} x_n = x$ in Y we also have $x \in X$. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ is also Cauchy considered as a sequence in Y, and $\lim_{n\to\infty} x_n = x$ exists since Y is complete. Then $x \in X$, and so X is complete.

(ii) $\Rightarrow$ (i)

Assume X complete. We must show that if $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence with $x_n \in X$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$ an element of Y, then we also have $x \in X$. Since $(x_n)_{n\in\mathbb{N}}$ is convergent in Y it is also Cauchy (in X and in Y). Then $x \in X$ since X is complete.