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8. Tutorial Analysis II for MCS
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(T8.1) Solution.

(i) We must show that the limit $\lim\limits_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_{a+\varepsilon}^{b} f$

exists. Now $\varphi : [\frac{1}{b-a}, \infty[\;\to [0, \infty[$

defined by

$$\varphi(t) = \int_{a+\frac{1}{t}}^{b} f$$

is increasing, since $f \geq 0$. Since $g \geq 0$ we also have $\int_{a+\frac{1}{t}}^{a_0} g \leq \int_{a}^{a_0} g$ for t s.t. $a + \frac{1}{t} < a_0$.

So

$$0 \leq \int_{a+\frac{1}{t}}^{b} f = \int_{a+\frac{1}{t}}^{a_0} f + \int_{a_0}^{b} f \leq \int_{a+\frac{1}{t}}^{a_0} g + \int_{a_0}^{b} f$$

$$\leq \int_{a}^{a_0} g + \int_{a_0}^{b} f \qquad \text{for } t \text{ s.t. } a + \frac{1}{t} < a_0,$$

and so φ is bounded. Hence $\lim_{t\to\infty} \varphi(t)$

exists by (T7.2), and

$$\lim_{t\to\infty} \varphi(t) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \varphi\left(\frac{1}{\varepsilon}\right) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_{a+\varepsilon}^{b} f.$$

(ii) We must show that the limit

$$\lim_{b\to\infty} \int_a^b f(x)\,dx$$ exists. Now

$\varphi : [a,\infty[\ \to [0,\infty[$ defined by

$\varphi(t) := \int_a^t f$ is increasing, since

$f \geq 0$. Since $g \geq 0$ we also have

$$\int_{a_0}^{t} g \leq \int_{a_0}^{\infty} g \quad \text{for } t \geq a_0.$$

Thus, for $t \in [a_0, \infty[$,

$$0 \leq \int_a^t f = \int_a^{a_0} f + \int_{a_0}^{t} f \leq \int_a^{a_0} f + \int_{a_0}^{t} g \leq \int_a^{a_0} f + \int_{a_0}^{\infty} g.$$

And so φ is bounded. Hence $\lim_{t\to\infty} \varphi(t)$ exists

by (T7.2), and $\lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} \int_a^t f$.

(T8.2) Solution.

(i) We have

$$t^{x-1} e^{-t} \le \frac{1}{t^{1-x}} \quad \text{for all } t>0,$$

hence $\int_0^1 t^{x-1} e^{-t}\,dt$ exists by (T8.1)(i)

(since $x>0$).

Furthermore, since $\lim_{t\to\infty} t^{x+1} e^{-t} = 0$

we get that there exists a $t_0 > 1$

such that

$$t^{x-1} e^{-t} \le \frac{1}{t^2} \quad \text{for } t \ge t_0.$$

Since $\int_1^\infty \frac{1}{t^2}\,dt$ exists it follows from (T8.1)(ii) that $\int_1^\infty t^{x-1}e^{-t}\,dt$ also exists. Then also

$$\int_0^\infty t^{x-1}e^{-t}\,dt \quad \text{exists.}$$

(ii) Integration by parts gives

$$\int_\varepsilon^R t^x e^{-t}\,dt = \Big[-t^x e^{-t}\Big]_{t=\varepsilon}^{t=R} + x\int_\varepsilon^R t^{x-1}e^{-t}\,dt,$$

for $0 < \varepsilon \leq R \in \mathbb{R}$.

Thus

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_\varepsilon^1 t^x e^{-t} = -1^x e^{-1} + x\int_0^1 t^{x-1}e^{-t}\,dt,$$

and

$$\lim_{R\to\infty} \int_1^R t^x e^{-t}\, dt = 1^x e^{-1} + x \int_1^\infty t^{x-1} e^{-t}\, dt.$$

So

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\, dt = x \int_0^\infty t^{x-1} e^{-t}\, dt =$$

$$x\,\Gamma(x).$$

Since $\Gamma(1) = \lim\limits_{R\to\infty} \int_0^R e^{-t}\, dt = \lim\limits_{R\to\infty} (1 - e^{-R}) = 1$,

$\overbrace{\text{we conclude}}^{\text{use induction to}}$ from $\Gamma(x+1) = x\Gamma(x)$ that

$\Gamma(n+1) = n!$.

That is, assume that $\Gamma(k+1) = k!$ for some $k \in \mathbb{N}_0$. Then $\Gamma(k+2) = (k+1)\,\Gamma(k+1) = (k+1)!$, and so $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$, since we already saw that $\Gamma(1) = 1$.