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7. Tutorial Analysis II MCS SS 2006

(T 7.1) Solution.

We assume that $f:[a,b]\to\mathbb{R}$ is integrable, and that $f(x)=g(x)$ for $x\notin\{a_1,\dots,a_n\}\subseteq[a,b]$.

We first show that g is bounded. Since f is integrable we know that it is bounded, so let $M\in\mathbb{R}$ be such that $M\geq|f(x)|$ for $x\in[a,b]$. Then

$$|g(x)|\leq\max\{M,|g(a_1)|,|g(a_2)|,\dots,|g(a_n)|\}$$

for all $x\in[a,b]$, so g is bounded.

Consider a step function $s:[a,b]\to\mathbb{R}$ with $s\leq f$, i.e. $s\in S_f$.

Define $s_1:[a,b]\to\mathbb{R}$ by

$$s_1(x):=\begin{cases} g(x) & \text{if } x\in\{a_1,\dots a_n\},\\ s(x) & \text{else.}\end{cases}$$

Then s_1 is a step function and $s_1(x) \le g(x)$ for all $x \in [a,b]$, so $s_1 \in S_g$.

Consider a $\underbrace{\text{partition } T = (x_0 = a < \ldots < x_p = b)}_{\text{associated with } s}$, and let $T' = (y_0 = a < \ldots < y_m = b)$ be a partition refining T and such that $\{a_1, \ldots, a_n\} \subseteq \{y_0, \ldots, y_m\}$.

Let $(\xi_i)_{i=1}^{m}$ be a system of intermediate points such that $\xi_i \in]y_{i-1}, y_i[$ for $1 \le i \le m$.

Then by definition

$$\int s = \sum_{i=1}^{m} s(\xi_i)(y_i - y_{i-1}),$$

$$\int s_1 = \sum_{i=1}^{m} s_1(\xi_i)(y_i - y_{i-1}),$$

so $\int s = \int s_1$, since $s(\xi_i) = s_1(\xi_i)$ for

$\xi_i \in]y_{i-1}, y_i[, \quad 1 \le i \le m.$

Since g is bounded we have that

$$\underline{\int} g = \sup\left\{ \int s : s \in S_g \right\} \text{ exists.}$$

Thus

$$\int f = \sup\left\{ \int s : s \in S_f \right\} \le \sup\left\{ \int s : s \in S_g \right\} = \underline{\int} g.$$

Consider now a step function $t : [a,b] \to \mathbb{R}$ with $f \le t$, i.e. $t \in S^f$.

Define $t_1 : [a,b] \to \mathbb{R}$ by

$$t_1(x) := \begin{cases} g(x) & \text{if } x \in \{a_1, \dots, a_n\}, \\ t(x) & \text{else.} \end{cases}$$

Then t_1 is a step function and $t_1(x) \ge g(x)$ for $x \in [a,b]$, so $t_1 \in S^g$.

Furthermore we can apply the same argument as above to conclude that $\int t = \int t_1$.

Since g is bounded we have that

$$\overline{\int} g = \inf \left\{ \int t : t \in S^g \right\} \text{ exists.}$$

Therefore

$$\int f = \inf \left\{ \int t : t \in S^f \right\} \geq \inf \left\{ \int t : t \in S^g \right\} = \overline{\int} g,$$

since

$$\left\{ \int t : t \in S^f \right\} \subseteq \left\{ \int t : t \in S^g \right\}.$$

Thus in total we have

$$\overline{\int} g \leq \int f \leq \underline{\int} g.$$

Since for all $s \in S_g$ and all $t \in S^g$ we have $s \leq t$, and thus $\int s \leq \int t$, we also get $\underline{\int} g \leq \overline{\int} g$.

Therefore $\underline{\int} g = \overline{\int} g$, and g is integrable with $\int g = \underline{\int} g = \overline{\int} g = \int f$.

This shows that if f is integrable, then g is integrable and $\int f = \int g$.

The same argument applies to show that if g is integrable, then f is integrable and $\int g = \int f$.

(T 7.2) Solution

(i) If $f : [a,\infty[\to [0,\infty[$ is not bounded, then for each $M \in [0,\infty[$ there exists $b \in [a,\infty[$ such that $f(b) > M$. Since f is increasing we conclude $f(x) > M$ for all $x \in [b,\infty[$.

We have by definition

$$\lim_{x\to\infty} f(x) = \infty \Leftrightarrow \lim_{\substack{t\to 0\\ t>0}} f\left(\tfrac{1}{t}\right) = \infty.$$

Let $M > 0$, and let $b \in [a,\infty[$ be such that $f(x) > M$ for $x \geq b$.

We can assume $b > 0$. Define

$\varphi :]0, \frac{1}{1+|a|}[\to [0,\infty[$ by $\varphi(t) := f(\frac{1}{t})$.

Then $\varphi(t) > M$ for all t in the domain of φ such that $t < \frac{1}{b}$. So

$$\lim_{\substack{t\to 0\\ t>0}} \varphi(t) = \infty, \text{ i.e. } \lim_{\substack{t\to 0\\ t>0}} f\left(\tfrac{1}{t}\right) = \infty.$$

Hence $\lim_{x\to\infty} f(x) = \infty$.

If f is bounded, then $\sup\{f(x) : x \in [a,\infty[\}$ exists, and $\|f\| = \sup\{f(x) : x \in [a,\infty[\}$.
Let $\varepsilon > 0$. By the characterization theorem for sups there exists $b \in [a,\infty[$ s.t.

$$\|f\| - \varepsilon < f(b) \leq \|f\|.$$

Then since f is increasing we have

$$\|f\| - \varepsilon < f(x) \leq \|f\| \quad \text{for all } x \in [b,\infty[,$$

so $\lim_{x\to\infty} f(x) = \|f\|$.

(ii) If f is not bounded then for each $M \in [0,\infty[$ there exists $b \in [a,\infty[$ s.t.

$f(x) > M$ for $x \geq b$. We can let $n \in \mathbb{N}$ be such that $n \geq b$, and get $f(n) > M$.
Thus $(f(n))_{n\geq a}$ is not bounded.
If on the other hand $(f(n))_{n\geq a}$ is not bounded, then for $M \in [0,\infty[$ we find $n \in \mathbb{N}$ with $n \geq a$ s.t. $f(n) > M$. Hence for

$M > 0$ there exists $x \in [a, \infty[$ such that $f(x) > M$, i.e. f is not bounded.