Fachbereich Mathematik
Dr. L. Leuştean
E. Briseid, S. Herrmann

TECHNISCHE UNIVERSITÄT DARMSTADT

23.05.2006

6. Tutorial Analysis II for MCS
Summer Term 2006

(T6.1)

Let $a < b \in \mathbb{R}$ and $C([a,b])$ be the set of all continuous functions $f : [a,b] \to \mathbb{R}$. For any $p \in \mathbb{R}, p \geq 1$, and for any $f \in C([a,b])$, define

$$\|f\|_p := \left(\int_a^b |f|^p \, dx\right)^{\frac{1}{p}}.$$

Prove that $\|\cdot\|_p$ is a norm on the set $C([a,b])$, that is, that for any $f, g \in C([a,b])$, and for any $\lambda \in \mathbb{R}$ the following hold:

(i) $\|f\|_p \geq 0$ and ($\|f\|_p = 0$ if and only if $f = 0$).

(ii) $\|\lambda f\|_p = |\lambda| \cdot \|f\|_p$.

(iii) $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Hint for (iii): Use the following intermediate steps:

(a) Let $a, b \geq 0$, and $p, q \in \mathbb{R}, p, q > 1$ be such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Prove the following inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1}$$

(b) Prove the **Hölder Inequality**: Let $p, q > 1$ be such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then for any functions $f, g \in C([a,b])$,

$$\int_a^b |fg| \, dx \leq \|f\|_p \cdot \|g\|_q. \tag{2}$$

(c) Prove the **Minkowski Inequality**: For any $p \geq 1$ and for any functions $f, g \in C([a,b])$,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \tag{3}$$

Solution.

(i) By Proposition 5.15, we have that $\|f\|_p = \left(\int_a^b |f|^p \, dx\right)^{\frac{1}{p}} \geq \left(\left|\int_a^b f^p \, dx\right|\right)^{\frac{1}{p}} \geq 0$. Obviously, if $f = 0$, then $\|f\|_p = 0$. It remains to prove the other implication, i.e., that $\|f\|_p = 0$ implies $f = 0$. Assume that $\|f\|_p = 0$, but $f \neq 0$. Then there is $c \in [a,b]$ such that $f(c) \neq 0$, so $|f|^p(c) > 0$. Since $|f|^p$ is continuous, there is an open neighborhood of c, let us say $U :=]c - \varepsilon, c + \varepsilon[\subseteq \mathbb{R}$, such that $|f|^p(x) > \dfrac{|f|^p(c)}{2}$ for all $x \in U \cap [a,b]$. It is easy to see that $U \cap [a,b]$ is an interval. Let $\alpha < \beta$ be its endpoints.

Let us now define the following function:

$$s : [a,b] \to \mathbb{R} \quad s(x) = \begin{cases} \dfrac{|f|^p(c)}{2} & \text{if } x \in U \cap [a,b] \\ 0, & \text{otherwise}. \end{cases}$$

Then s is a step function and $s \leq |f|^p$, so

$$\int_a^b |f|^p \, dx \geq \int_a^b s = \frac{|f|^p(c)}{2}(\beta - \alpha) > 0.$$

Hence

$$\|f\|_p = \left(\int_a^b |f|^p \, dx\right)^{\frac{1}{p}} > 0,$$

which is a contradiction.

(ii) Applying Proposition 5.14 (ii), we get

$$\begin{aligned} \|\lambda f\|_p &= \left(\int_a^b |\lambda f|^p \, dx\right)^{\frac{1}{p}} = \left(\int_a^b |\lambda|^p |f|^p \, dx\right)^{\frac{1}{p}} = \left(|\lambda|^p \int_a^b |f|^p \, dx\right)^{\frac{1}{p}} \\ &= |\lambda| \left(\int_a^b |f|^p \, dx\right)^{\frac{1}{p}} = |\lambda| \cdot \|f\|_p. \end{aligned}$$

(iii) (a) If either $a = 0$ or $b = 0$, then the inequality is obvious. If $a, b > 0$, then we can apply the arithmetical-geometrical inequality from (G3.2) (ii) with $\alpha := \frac{1}{p}$, $\beta := \frac{1}{q}$, $a := a^p$ and $b := b^q$.

(b) If either $\|f\|_p = 0$ or $\|g\|_q = 0$, then by (i) the inequality is trivial. Assume that $\|f\|_p, \|g\|_q > 0$. Let $f_1 := \dfrac{f}{\|f\|_p}, g_1 := \dfrac{g}{\|g\|_q}$. Now $\|f_1\|_p = \|g_1\|_q = 1$. Taking $a := |f_1(x)|, b := |g_1(x)|$ in (a), for $x \in [a,b]$, we get

$$|f_1 g_1| \leq \frac{|f_1|^p}{p} + \frac{|g_1|^q}{q}.$$

So

$$\begin{aligned}\int_a^b |f_1 g_1|\,dx &\leq \int_a^b \left(\frac{|f_1|^p}{p}+\frac{|g_1|^q}{q}\right)dx = \int_a^b \frac{|f_1|^p}{p}\,dx + \int_a^b \frac{|g_1|^q}{q}\,dx \\ &= \frac{1}{p}\left(\|f_1\|_p\right)^p + \frac{1}{q}\left(\|g_1\|_q\right)^q = \frac{1}{p}+\frac{1}{q} = 1.\end{aligned}$$

Since $\int_a^b |f_1 g_1|\,dx = \int_a^b \frac{|fg|}{\|f\|_p\|g\|_q}\,dx = \frac{1}{\|f\|_p\|g\|_q}\int_a^b |fg|\,dx$, we get that

$$\frac{1}{\|f\|_p\|g\|_q}\int_a^b |fg|\,dx \leq 1,$$

that is,

$$\int_a^b |fg|\,dx \leq \|f\|_p\|g\|_q.$$

(c) For the case $p=1$ we have

$$\|f+g\|_1 = \int_a^b |f+g| \leq \int_a^b (|f|+|g|) = \int_a^b |f| + \int_a^b |g| = \|f\|_1 + \|g\|_1.$$

Also the case $\|f+g\|_p = 0$ is trivial, so assume that $p>1$ and $\|f+g\|_p \neq 0$. Then

$$\begin{aligned}(\|f+g\|_p)^p &= \int_a^b |f+g|^p\,dx = \int_a^b |f+g|^{p-1}|f+g|\,dx \\ &\leq \int_a^b |f+g|^{p-1}(|f|+|g|)\,dx \\ &= \int_a^b |f+g|^{p-1}|f|\,dx + \int_a^b |f+g|^{p-1}|g|\,dx.\end{aligned}$$

Applying (b) with $q := \frac{p}{p-1}$, we get

$$\int_a^b |f+g|^{p-1}|f|\,dx \leq \|f\|_p \| |f+g|^{p-1}\|_q,$$
$$\int_a^b |f+g|^{p-1}|g|\,dx \leq \|g\|_p \| |f+g|^{p-1}\|_q,$$

so

$$(\|f+g\|_p)^p \leq (\|f\|_p + \|g\|_p)\| |f+g|^{p-1}\|_q. \tag{4}$$

Since

$$\| |f+g|^{p-1}\|_q = \left(\int_a^b \left(|f+g|^{p-1}\right)^q\right)^{\frac{1}{q}} = \left(\int_a^b |f+g|^p\,dx\right)^{\frac{p-1}{p}} = (\|f+g\|_p)^{p-1},$$

(4) becomes

$$(\|f+g\|_p)^p \leq (\|f\|_p + \|g\|_p)\,(\|f+g\|_p)^{p-1}.$$

That is,

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

■

Orientation Colloquium

The Department of Mathematics' Research Groups present themselves.

Monday, 29.05.2006 – 16:15-17:15 – S207/109

Prof. Dr. Burkhard Kümmerer

FG Algebra, Geometrie und Funktionalanalysis

"Im Dreiländereck Funktionalanalysis – Stochastik – Mathematische Physik"

After the talk there will be a relaxed get-together (coffee, tea and biscuits) in S215/219, where interested people can discuss the talk and become more acquainted with the lecturer.