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16.05.2006

5. Tutorial Analysis II for MCS
Summer Term 2006

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. For any partition $P = (a = x_0 < x_1 < \ldots < x_n = b)$ of $[a, b]$ define

$$\|P\| := \max\{x_i - x_{i-1} : 1 \le i \le n\},$$
$$\sigma(P, f, \xi) := \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}), \qquad \text{where } \xi_i \in [x_{i-1}, x_i], \quad 1 \le i \le n.$$

The sums $\sigma(P, f, \xi)$ are called the *Riemann sums* associated with the function f, the partition P, and the system of intermediate points $\xi = (\xi_i)_{i=1}^n = (\xi_1, \xi_2, \ldots, \xi_n)$.

(T5.1)

(i) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Prove that the following are equivalent.

(1) f is integrable.

(2) There is an $I \in \mathbb{R}$ such that for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\left| \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - I \right| < \varepsilon$$

for any partition $P = (a = x_0 < x_1 < \ldots x_n = b)$ of $[a, b]$ with $\|P\| < \delta$ and for every choice of points $\xi_1, \ldots, \xi_n$ with $\xi_i \in [x_{i-1}, x_i]$ for $1 \le i \le n$.

In case such an $I \in \mathbb{R}$ as specified above exists, it is the integral of f. Riemann defined the integral of $f : [a, b] \to \mathbb{R}$ as outlined in this exercise, rather than the way it is done in Hofmann's book.

Hint: For (1) $\Rightarrow$ (2), argue that there exist step functions s, t with $s \le f \le t$ and $\int t - \int s < \varepsilon/2$ and associated partition $P' = (a = y_0 < y_1 < \ldots < y_m = b)$. Now consider a partition $P = (a = x_0 < x_1 < \ldots < x_n = b)$ and let $\xi = (\xi)_{i=1}^n$ be a choice of points with $\xi_i \in [x_{i-1}, x_i]$ for $1 \le i \le n$. Consider the union Π of those intervals $]x_{i-1}, x_i[$ such that there exists a $1 \le j \le m$ with $[x_{i-1}, x_i] \subseteq]y_{j-1}, y_j[$. Notice that there can be at most $2m$ intervals $]x_{i-1}, x_i[$ such that $]x_{i-1}, x_i[\not\subseteq \Pi$.

(ii) Let $f : [a, b] \to \mathbb{R}$ be integrable. Prove that for every sequence $(P_n)_{n\in\mathbb{N}}$ of partitions $P_n = (a = x_0^{(n)} < x_1^{(n)} < \ldots < x_{p_n}^{(n)} = b)$ with $\lim_{n\to\infty} \|P_n\| = 0$, and for all systems $\xi^{(n)} = (\xi_i^{(n)})_{i=1}^{p_n}$ of intermediate points with $\xi_i^{(n)} \in [x_{i-1}^{(n)}, x_i^{(n)}]$ $(n \in \mathbb{N}, 1 \le i \le p_n)$,

$$\lim_{n\to\infty} \sigma(P_n, f, \xi^{(n)}) = \int_a^b f dx.$$

Solution.

(i) (1) $\Rightarrow$ (2) : Let $\varepsilon > 0$. Since f is integrable there exist step functions s, t with $s \le f \le t$ and

$$\int t - \int s < \frac{\varepsilon}{2}.$$

Since f is bounded we can define

$$M := \sup\{|f(x)| : x \in [a, b]\}.$$

Let $P' = (a = y_0 < y_1 < \ldots < y_m = b)$ be a partition associated with s and t. Let now

$$\delta = \frac{\varepsilon}{8Mm}.$$

Let $P = (a = x_0 < x_1 < \ldots < x_n = b)$ be a partition such that $\|P\| < \delta$, and let $\xi = (\xi)_{i=1}^n$ be a choice of points with $\xi_i \in [x_{i-1}, x_i]$ for $1 \le i \le n$. We define the step function $F \in S[a, b]$ by $F(a) := 0$ and by for each $1 \le i \le n$ letting $F(x_i) := 0$ and $F(x) := f(\xi_i)$ for $x_{i-1} < x < x_i$. Then

$$\int F = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}).$$

We have

$$s(x) - 2M \le F(x) \le t(x) + 2M$$

for all $x \in [a, b]$. Furthermore, if $[x_{i-1}, x_i] \subseteq]y_{j-1}, y_j[$ for some $1 \le j \le m$, then we have

$$s(x) \le F(x) \le t(x)$$

for all $x \in]x_{i-1}, x_i[$. Let $\Pi \subseteq [a, b]$ be the union of those intervals $]x_{i-1}, x_i[$ such that there exists a $1 \le j \le m$ with $[x_{i-1}, x_i] \subseteq]y_{j-1}, y_j[$. Notice that there can be at most $2m$ intervals $]x_{i-1}, x_i[$ such that $]x_{i-1}, x_i[\not\subseteq \Pi$. We define the step function $\phi : [a, b] \to \mathbb{R}$ by $\phi(x) := 0$ if $x \in \Pi$ and $\phi(x) := 2M$ if $x \notin \Pi$. Then by the above we have

$$s(x) - \phi(x) \le F(x) \le t(x) + \phi(x)$$

for all $x \in [a, b]$. Since there are at most $2m$ intervals $]x_{i-1}, x_i[$ on which ϕ is not constant 0, we get

$$\int \phi \le 2M(2m\delta) = \varepsilon/2.$$

So

$$\int s - \varepsilon/2 \leq \int F \leq \int t + \varepsilon/2.$$

By our choice of s, t we have

$$\int f < \int s + \varepsilon/2$$

and

$$\int t < \int f + \varepsilon/2.$$

Thus

$$\left|\int f - \int F\right| < \varepsilon,$$

as we wanted to show.

(2) $\Rightarrow$ (1) : Let $\varepsilon > 0$, and let δ be such that

$$\left|\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - I\right| < \varepsilon/4$$

for any partition $P = (a = x_0 < x_1 < \ldots x_n = b)$ of $[a, b]$ with $\|P\| < \delta$ and for every choice of points $\xi_1, \ldots, \xi_n$ with $\xi_i \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$. Let now $P = (a = x_0 < x_1 < \ldots x_n = b)$ be a partition with $\|P\| < \delta$ and $(b - a)/n < \varepsilon/4$.

For any $1 \leq i \leq n$ we let

$$m_i := \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \quad M_i := \sup\{f(x) : x_{i-1} \leq x \leq x_i\}.$$

Define now $s_1, t_1 : [a, b] \to \mathbb{R}$ by:

$$s_1(x) = m_i, \quad t_1(x) = M_i \quad \text{for all } x \in [x_{i-1}, x_i[, \ 1 \leq i \leq n, \quad s_1(b) = t_1(b) = f(b).$$

Then s_1, t_1 are step functions, $s_1 \leq f \leq t_1$, and

$$\int t_1 - \int s_1 = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) - \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

Pick now $\xi_1, \ldots, \xi_n$ such that $\xi_i \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$ and such that

$$M_i - f(\xi_i) < 1/n.$$

Then

$$\int t_1 = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) < \sum_{i=1}^{n} (f(\xi_i) + 1/n)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) + \frac{b-a}{n}.$$

So

$$\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \leq \int t_1 < \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) + \frac{b-a}{n},$$

and furthermore

$$\left|\int t_1 - I\right| < \varepsilon/2,$$

since $(b - a)/n < \varepsilon/4$.

Let now $\xi'_1, \ldots, \xi'_n$ be such that $\xi'_i \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$ and such that

$$f(\xi'_i) - m_i < 1/n.$$

Then

$$\int s_1 = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) > \sum_{i=1}^{n} (f(\xi'_i) - 1/n)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(\xi'_i)(x_i - x_{i-1}) - \frac{b-a}{n}.$$

So

$$\sum_{i=1}^{n} f(\xi'_i)(x_i - x_{i-1}) \geq \int s_1 > \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - \frac{b-a}{n},$$

and furthermore

$$\left|\int s_1 - I\right| < \varepsilon/2,$$

since $(b - a)/n < \varepsilon/4$.

Hence

$$\int t_1 - \int s_1 < \varepsilon,$$

and we can apply the Riemann Criterion (Theorem 5.12) to conclude that f is integrable.

(ii) This follows easily from (i). Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of partitions $P_n = (a = x_0^{(n)} < x_1^{(n)} < \ldots < x_{p_n}^{(n)} = b)$ with $\lim_{n\to\infty} \|P_n\| = 0$, and let $\xi^{(n)} = (\xi_i^{(n)})_{i=1}^{p_n}$ be systems of intermediate points with $\xi_i^{(n)} \in [x_{i-1}^{(n)}, x_i^{(n)}]$ $(n \in \mathbb{N}, 1 \leq i \leq p_n)$. Let $\varepsilon > 0$, and let δ be as in (i). Since $\lim_{n\to\infty} \|P_n\| = 0$, there is $N > 0$ such that $\|P_n\| < \delta$ for any $n \geq N$. Applying (i), it follows that for any $n \geq N$,

$$\left|\sigma(P_n, f, \xi^{(n)}) - \int_a^b f\right| = \left|\sum_{i=1}^{p_n} f(\xi_i^{(n)})(x_i^{(n)} - x_{i-1}^{(n)}) - \int_a^b f\right| < \varepsilon.$$

Hence,

$$\lim_{n\to\infty} \sigma(P_n, f, \xi^{(n)}) = \int_a^b f dx.$$

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(T5.2) Let $a < b \in \mathbb{R}$, and let $\exp : [a, b] \to \mathbb{R}$ be the exponential function. Use (T5.1)(ii) to compute its integral.

Solution. Since exp is monotone it is integrable. We apply (T5.1)(ii) to compute the integral. For every $n \in \mathbb{N}$, we define the partition $P_n = (a = x_0^{(n)} < x_1^{(n)} < \ldots < x_n^{(n)} = b)$, where $x_i^{(n)} = a + i \cdot \dfrac{b-a}{n}$ for all $0 \leq i \leq n$. Then $(P_n)_{n\in\mathbb{N}}$ is a sequence of partitions of $[a, b]$ satisfying

$$\lim_{n\to\infty} \|P_n\| = \lim_{n\to\infty} \frac{b-a}{n} = 0.$$

Consider now the systems $\xi^{(n)} = (\xi_i^{(n)})_{i=1}^n$ of intermediate points with $\xi_i^{(n)} = x_{i-1}^{(n)}$ for $1 \leq i \leq n$. Then

$$\begin{aligned}
\sigma(P_n, f, \xi^{(n)}) &= \sum_{i=1}^n f(\xi_i^{(n)})(x_i^{(n)} - x_{i-1}^{(n)}) = \sum_{i=1}^n f\left(a + (i-1)\cdot\frac{b-a}{n}\right)\cdot\frac{b-a}{n} \\
&= \frac{b-a}{n}\sum_{i=1}^n \exp\left(a + (i-1)\cdot\frac{b-a}{n}\right) \\
&= \frac{b-a}{n}\exp(a)\sum_{i=1}^n \left[\exp\left(\frac{b-a}{n}\right)\right]^{i-1} \\
&= \frac{b-a}{n}\exp(a)\frac{\left[\exp\left(\frac{b-a}{n}\right)\right]^n - 1}{\exp\left(\frac{b-a}{n}\right) - 1}. \\
&= \exp(a)(\exp(b-a) - 1)\frac{\frac{b-a}{n}}{\exp\left(\frac{b-a}{n}\right) - 1} \\
&= (\exp(b) - \exp(a))\,\frac{\frac{b-a}{n}}{\exp\left(\frac{b-a}{n}\right) - 1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\lim_{n\to\infty}\sigma(P_n, f, \xi^{(n)}) &= (\exp(b) - \exp(a))\lim_{n\to\infty}\frac{\frac{b-a}{n}}{\exp\left(\frac{b-a}{n}\right) - 1} = (\exp(b) - \exp(a))\cdot 1 \\
&= \exp(b) - \exp(a),
\end{aligned}$$

since $\lim\limits_{\substack{x\to 0\\ x\neq 0}} \dfrac{x}{\exp(x) - 1} = 0$ by the Rule of Bernoulli and de l'Hôpital.

Using now (T5.1)(ii), we get that

$$\int_a^b f dx = \lim_{n\to\infty}\sigma(P_n, f, \xi^{(n)}) = \exp(b) - \exp(a).$$

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