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4. Tutorial Analysis II for MCS
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(T4.1) Solution.

(i) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of reals. We write $\lim\limits_{n\to\infty} x_n = -\infty$ if

$$(\forall M\in\mathbb{R})(\exists m\in\mathbb{N})(\forall n\geq m)(x_n < M).$$

(ii) Let a be an accumulation point of a subset X of a metric space, and let $f: X\to\mathbb{R}$. We write $\lim\limits_{x\to a} f(x) = -\infty$ if

$$(\forall M\in\mathbb{R})(\exists\delta>0)(\forall x\in X)(d(x,a)<\delta \Rightarrow f(x)<M).$$

(iii) Let $X\subseteq\mathbb{R}$, $f: X\to\mathbb{R}$ and let $a\in\mathbb{R}$ be an accumulation point of $X\cap\,]-\infty,a[$.

We write $\lim\limits_{\substack{x\to a\\ x<a}} f(x) = \infty$ if

$$(\forall M\in\mathbb{R})(\exists\delta>0)(\forall x\in X\cap\,]-\infty,a[)(d(x,a)<\delta \Rightarrow f(x)>M).$$

(IV). Let $a \in \mathbb{R}$ and $f :]-\infty, a[\to \mathbb{R}$. We write $\lim\limits_{x \to -\infty} f(x) = \ell$, for some $\ell \in \mathbb{R}$, if

$$\lim_{\substack{y \to 0 \\ y < 0}} f\left(\tfrac{1}{y}\right) = \ell.$$

We write $\lim\limits_{x \to -\infty} f(x) = \infty$ if $\lim\limits_{\substack{y \to 0 \\ y<0}} f\left(\tfrac{1}{y}\right) = \infty$.

~~(I.4.2)~~

(T4.2)

(i). Since $\lim_{\substack{x\to a\\ x>a}} g'(x)\neq 0$ and $\lim_{\substack{x\to a\\ x>a}} g(x)=\lim_{\substack{x\to a\\ x>a}} f(x)=\infty$

we can consider an interval $]a,a+\delta[$ s.t.

$g'(x)\neq 0$ and $g(x)>1$ for $x\in]a,a+\delta[$.

We can furthermore assume that for $x\in]a,a+\delta[$ there exists a $y>a$ with $f(y)\leq\sqrt{f(x)}$ and $g(y)\leq\sqrt{g(x)}$.

Then we can define $u(x):=\inf\{y>a: f(y)\leq\sqrt{f(x)} \wedge g(y)\leq\sqrt{g(x)}\}$.

Since f and g are continuous we get $f(u(x))\leq\sqrt{f(x)}$ and $g(u(x))\leq\sqrt{g(x)}$, with equality in at least one of the cases.

Therefore $\lim_{\substack{x\to a\\ x>a}} u(x)=a$. According to the generalized mean value theorem there exists a $c(x)$, located properly between x and $u(x)$, s.t.

$$\frac{f'(c(x))}{g'(c(x))}=\frac{f(x)-f(u(x))}{g(x)-g(u(x))}.$$

(Here $c(x)$ depends also on $u(x)$.)

Therefore, $\lim\limits_{\substack{x \to a \\ x > a}} \dfrac{f(x) - f(u(x))}{g(x) - g(u(x))} = \lim\limits_{\substack{x \to a \\ x > a}} \dfrac{f'(c(x))}{g'(c(x))}$

$$= \lim_{\substack{x \to a \\ x > a}} \frac{f'(x)}{g'(x)} = \ell .$$

On the other hand,

$$\lim_{\substack{x \to a \\ x > a}} \frac{f(x) - f(u(x))}{g(x) - g(u(x))} = \lim_{\substack{x \to a \\ x > a}} \frac{f(x)}{g(x)} \cdot \frac{1 - \dfrac{f(u(x))}{f(x)}}{1 - \dfrac{g(u(x))}{g(x)}} .$$

And since $f(u(x)) \le \sqrt{f(x)}$ and $g(u(x)) \le \sqrt{g(x)}$,

we can conclude that

$$\lim_{\substack{x \to a \\ x > a}} \frac{f(x)}{g(x)} \cdot \frac{1 - \dfrac{f(u(x))}{f(x)}}{1 - \dfrac{g(u(x))}{g(x)}} = \lim_{\substack{x \to a \\ x > a}} \frac{f(x)}{g(x)} ,$$

since the former limit exists. So

$$\lim_{\substack{x \to a \\ x > a}} \frac{f(x)}{g(x)} = \ell .$$

(ii) We have

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{\substack{x\to 0\\ x>0}} \frac{f'(\frac{1}{x})}{g'(\frac{1}{x})} = \lim_{\substack{x\to 0\\ x>0}} \frac{f'(\frac{1}{x})\cdot(-x^{-2})}{g'(\frac{1}{x})\cdot(-x^{-2})}$$

Since furthermore $h_1, h_2 :]0, \frac{1}{1+|a|}] \to \mathbb{R}$ defined by $h_1(x) := f(\frac{1}{x})$, $h_2(x) := g(\frac{1}{x})$ are differentiable and such that

$$\lim_{\substack{x\to 0\\ x>0}} h_1(x) = \lim_{\substack{x\to 0\\ x>0}} h_2(x) = \infty, \text{ and } \lim_{\substack{x\to 0\\ x>0}} h_2'(x) \neq 0,$$

we can use the result in (i) above to conclude

$$\lim_{\substack{x\to 0\\ x>0}} \frac{h_1(x)}{h_2(x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = \ell,$$

since

$$\lim_{\substack{x\to 0\\ x>0}} \frac{h_1'(x)}{h_2'(x)} = \lim_{\substack{x\to 0\\ x>0}} \frac{f'(\frac{1}{x})\cdot(-x^{-2})}{g'(\frac{1}{x})\cdot(-x^{-2})} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

(T.4.3) Solution.

(i) Applying the result of (T4.2) (ii) n times, we obtain

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{n x^{n-1}}{e^x} = \ldots = \lim_{x\to\infty} \frac{n!}{e^x} = 0.$$

(ii) Both P and the exponential function are n times differentiable, and we have

$$\lim_{x\to\infty} P^{(i)}(x) = \infty \quad \text{for } 0 \le i \le n-1.$$

Then the rule of l'Hôpital gives us

$$\lim_{x\to\infty} \frac{P(x)}{e^x} = \lim_{x\to\infty} \frac{P'(x)}{e^x} = \ldots = \lim_{x\to\infty} \frac{P^{(n)}(x)}{e^x} = 0,$$

since $P^{(n)}$ is a constant function.