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2. Tutorial Analysis I for MCS
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(T2.1)

Let $f : I \to \mathbb{R}$ be a function of class C^n on an interval $I \subseteq \mathbb{R}$ and let a be an inner point of I. Assume that

$$f'(a) = f''(a) = \ldots = f^{(n-1)}(a) = 0, f^{(n)}(a) \neq 0.$$

Prove the following assertions:

(i) If n is even, then f has a local extremum at a. More precisely, if $f^{(n)}(a) > 0$, then f has a local minimum at a, if $f^{(n)}(a) < 0$, then f has a local maximum at a.

(ii) If n is odd, then f does not have a local extremum at a.

Solution. Since $f^{(n)}$ is continuous, a is an inner point of I, and $f^{(n)}(a) \neq 0$, there is a $\delta > 0$ such that $]a-\delta, a+\delta[\subseteq I$ and such that $f^{(n)}$ has the same sign as $f^{(n)}(a)$ on $]a-\delta, a+\delta[$. Let $x \in]a-\delta, a+\delta[$. Applying Taylor's Theorem and using the hypothesis, we get that there is a u located properly between a and x such that

$$f(x) - f(a) = \frac{f^{(n)}(u)}{n!}(x-a)^n.$$

(i) Assume that n is even, so $(x-a)^n \geq 0$. If $f^{(n)}(a) > 0$, then $f^{(n)}(u) > 0$, and it follows that $f(x) - f(a) \geq 0$ for all $x \in]a-\delta, a+\delta[$. Thus, f attains a local minimum at a. Similarly, if $f^{(n)}(a) < 0$, then $f(x) - f(a) \leq 0$ for all $x \in]a-\delta, a+\delta[$, so f has a local maximum at a.

(ii) Assume now that n is odd. Then $(x-a)^n > 0$ for $x > a$, and $(x-a)^n < 0$ for $x < a$. It follows that if $f^{(n)}(a) > 0$, then $f(x) - f(a) > 0$ for $x \in]a, a+\delta[$, and $f(x) - f(a) < 0$ for $x \in]a-\delta, a[$. Hence, f does not have a local extremum at a. Similarly for $f^{(n)}(a) < 0$. ∎

(T2.2)

Prove Corollary 4.12:
Assume that the function $f : U_\rho(0) \to \mathbb{K}$ satisfies the hypotheses of Theorem 4.11. (Here $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$.) Then all successive derivatives $f^{(k)} : U_\rho(0) \to \mathbb{K}$ exist (recall that $f^{(0)} = f$ and $f^{(k+1)} = (f^{(k)})'$) and

$$f^{(k)}(x) = k! \sum_{n=k}^{\infty} \binom{n}{k} a_n x^{n-k} = k! \sum_{n=0}^{\infty} \binom{n+k}{k} a_{n+k} x^n.$$

Solution. Recall that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

We prove by induction on k that $f^{(k)}$ is a convergent power series with radius of convergence ρ satisfying

$$f^{(k)}(x) = k! \sum_{n=0}^{\infty} \binom{n+k}{k} a_{n+k} x^n.$$

Induction start, i.e. case $k = 0$:

$$f^{(0)}(x) = f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Induction step:
Let $k \geq 0$, and define

$$a'_n := k! \binom{n+k}{k} a_{n+k}.$$

By the induction hypothesis

$$f^{(k)}(x) = \sum_{n=0}^{\infty} a'_n x^n$$

has radius of convergence ρ. By (the proof of) 4.11 we obtain that

$$f^{(k+1)}(x) = (f^{(k)})'(x) = \sum_{n=0}^{\infty} (n+1) a'_{n+1} x^n$$

has radius of convergence ρ. Furthermore

$$(n+1)k! \binom{n+1+k}{k} = (n+1)k! \binom{n+(k+1)}{k+1} \left(\frac{k+1}{n+1}\right) = (k+1)! \binom{n+(k+1)}{k+1}.$$

Hence

$$f^{(k+1)}(x) = (k+1)! \sum_{n=0}^{\infty} \binom{n+(k+1)}{k+1} a_{n+(k+1)} x^n.$$

This concludes the proof. ∎

(T2.3) Supplementary exercise.

Prove that if $I \subseteq \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is differentiable, then the image $f'(I)$ of the derivative is an interval.

Solution. See the sketch of a proof of Theorem 4.31 in Hofmann. ■