Fachbereich Mathematik
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20.04.2006

1. Tutorial Analysis I for MCS
Winter Term 2005/2006

(T1.1) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that if the derivative of f has at most k distinct zeros, then f has at most $k+1$ distinct zeros.

Hint: Use Rolle's Theorem.

Solution. Assume that f has $k+2$ distinct zeros $x_1 < x_2 < \ldots < x_{k+1} < x_{k+2}$. Then, by applying Rolle's Theorem to f on the $k+1$ intervals $[x_1, x_2], \ldots, [x_{k+1}, x_{k+2}]$, we get $y_1 \in]x_1, x_2[, \ldots, y_{k+1} \in]x_{k+1}, x_{k+2}[$ such that $f'(y_1) = \ldots = f'(y_{k+1}) = 0$. Thus, f' has $k+1$ distinct zeros, which contradicts the hypothesis. ■

(T1.2) Let $A \subseteq \mathbb{R}$ be an open set and let $f : A \to \mathbb{R}$ be a continuous function, which is differentiable on $A \setminus \{x_0\}$ for some $x_0 \in A$. Prove that if $\lim\limits_{\substack{x \to x_0 \\ x \neq x_0}} f'(x)$ exists, then f is differentiable at x_0 and $f'(x_0) = \lim\limits_{\substack{x \to x_0 \\ x \neq x_0}} f'(x)$.

Solution. Let $\lambda := \lim\limits_{\substack{x \to x_0 \\ x \neq x_0}} f'(x)$. We have to prove that

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{f(x) - f(x_0)}{x - x_0} = \lambda.$$

By hypothesis, A is open, so there is an $\varepsilon > 0$ such that $]x_0 - \varepsilon, x_0 + \varepsilon[\subseteq A$. Since we are interested only in the limit $x \to x_0, x \neq x_0$, we may assume that $x \in]x_0 - \varepsilon, x_0 + \varepsilon[$.
If $x < x_0$, then f is differentiable on $]x, x_0[$, and continuous on $[x, x_0]$ so we can apply the Mean Value Theorem to get $\xi(x) \in]x, x_0[$ such that

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(\xi(x)).$$

Since $\lim\limits_{\substack{x \to x_0 \\ x < x_0}} \xi(x) = x_0$, we conclude that $\lim\limits_{\substack{x \to x_0 \\ x < x_0}} f'(\xi(x)) = \lim\limits_{\substack{x \to x_0 \\ x < x_0}} f'(x) = \lambda$, so

$$D^-_{x_0} f := \lim_{\substack{x \to x_0 \\ x < x_0}} \frac{f(x) - f(x_0)}{x - x_0} = \lambda.$$

We get similarly that

$$D^+_{x_0} f := \lim_{\substack{x \to x_0 \\ x > x_0}} \frac{f(x) - f(x_0)}{x - x_0} = \lambda.$$

Hence f is differentiable at x_0 and $f'(x_0) = \lambda$. ■