Fachbereich Mathematik
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# 13. Home work Analysis II for MCS
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**(H13.1)**

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) = (e^x \cos y, e^x \sin y).$$

(i) Show that $f$ is differentiable and compute its derivative.

(ii) Show that $f$ is locally invertible around every point in $\mathbb{R}^2$.

(iii) Show that $f$ does not have a global inverse.

**Solution.** Handwritten.

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**(H13.2)**

Let us consider the following system of equations:

$$\begin{aligned} x^2 + 4y^2 + 9z^2 &= 1 \\ x + y + z &= 0. \end{aligned}$$

(i) Show that this system can be solved uniquely with respect to $y$ and $z$,

$$\begin{pmatrix} y \\ z \end{pmatrix} = g(x),$$

in a neighborhood of the point $(0, 1/\sqrt{13}, -1/\sqrt{13})$.

(ii) Compute $g'(0)$.

**Solution.**



(i) Let

$$f : \mathbb{R}^3 \to \mathbb{R}^2,\ f(x, y, z) = \begin{pmatrix} x^2 + 4y^2 + 9z^2 - 1 \\ x + y + z \end{pmatrix}.$$

We have to prove that the equation $f(x, y, z) = 0$ can be solved uniquely with respect to $y$ and $z$ for $(x, y, z)$ near $(0, 1/\sqrt{13}, -1/\sqrt{13})$. We apply the Implicit Function Theorem. First, let us remark that $f(0, 1/\sqrt{13}, -1/\sqrt{13}) = 0$. Furthermore

$$J_f(x, y, z) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x, y, z) & \frac{\partial f_1}{\partial y}(x, y, z) & \frac{\partial f_1}{\partial z}(x, y, z) \\ \frac{\partial f_2}{\partial x}(x, y, z) & \frac{\partial f_2}{\partial y}(x, y, z) & \frac{\partial f_2}{\partial z}(x, y, z) \end{pmatrix} = \begin{pmatrix} 2x & 8y & 18z \\ 1 & 1 & 1 \end{pmatrix}.$$

Since all the partial derivatives are continuous on $\mathbb{R}^3$ it follows by Proposition 9.41 that $f$ is continuously differentiable on $\mathbb{R}^3$.

We must test whether the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial y}(x, y, z) & \frac{\partial f_1}{\partial z}(x, y, z) \\ \frac{\partial f_2}{\partial y}(x, y, z) & \frac{\partial f_2}{\partial z}(x, y, z) \end{pmatrix} = \begin{pmatrix} 8y & 18z \\ 1 & 1 \end{pmatrix}$$

is invertible at $(0, 1/\sqrt{13}, -1/\sqrt{13})$. At this point, the determinant of the matrix is

$$\begin{vmatrix} 8\frac{1}{\sqrt{13}} & -18\frac{1}{\sqrt{13}} \\ 1 & 1 \end{vmatrix} = \frac{26}{\sqrt{13}} \neq 0,$$

hence the matrix is invertible.

Consequently, there exist open neighborhoods $U = ]-\delta, \delta[$ of 0 in $\mathbb{R}$, respectively $V$ of $(1/\sqrt{13}, -1/\sqrt{13})$ in $\mathbb{R}^2$, and a continuously differentiable function $g : U \to V, g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ such that $g(0) = \begin{pmatrix} 1/\sqrt{13} \\ -1/\sqrt{13} \end{pmatrix}$, $f(x, g_1(x), g_2(x)) = 0$ for all $x \in ]-\delta, \delta[$, and moreover $g(x)$ is the unique solution of the equation $f(x, y, z) = 0$ with $x \in ]-\delta, \delta[$, $(y, z) \in V$.

(ii) We have $f_i(x, g_1(x), g_2(x)) = 0$, $i = 1, 2$, for $x \in ]-\delta, \delta[$. We define $h : ]-\delta, \delta[ \to \mathbb{R}$ by $h(x) := f_1(x, g_1(x), g_2(x)) = x^2 + 4(g_1(x))^2 + 9(g_2(x))^2 - 1$. Then $h$ is differentiable, with $h'(x) = 2x + 8g_1(x)g_1'(x) + 18g_2(x)g_2'(x) = 0$. Likewise we conclude from $g_1(x) + g_2(x) = -x$ that $g_1'(x) + g_2'(x) = -1$ for $x \in ]-\delta, \delta[$. Thus for $x \in ]-\delta, \delta[$ we have

$$\begin{aligned} 2x + 8g_1(x)g_1'(x) + 18g_2(x)g_2'(x) &= 0, \\ 1 + g_1'(x) + g_2'(x) &= 0. \end{aligned}$$

Inserting $g_1'(x) = -1 - g_2'(x)$ into the first equation gives $2x - 8g_1(x) - 8g_1(x)g_2'(x) + 18g_2(x)g_2'(x) = 0$, and hence

$$g_2'(x) = \frac{4g_1(x) - x}{9g_2(x) - 4g_1(x)}, \quad g_1'(x) = \frac{-9g_2(x) + x}{9g_2(x) - 4g_1(x)}.$$

In particular, $g_1'(0) = -9/13$, $g_2'(0) = -4/13$. Thus,

$$g'(0) = \begin{pmatrix} -9/13 \\ -4/13 \end{pmatrix}.$$

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**(H13.3)**

Let $a \in \mathbb{R}$, and define $h_1, h_2 : \mathbb{R}^2 \to \mathbb{R}$ by

$$h_1(x, y) = x \cos a - y \sin a$$

and

$$h_2(x, y) = x \sin a + y \cos a.$$

We will write $u = h_1(x, y)$ and $v = h_2(x, y)$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function from $\mathbb{R}^2$ into $\mathbb{R}$, and let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x, y) = f(h_1(x, y), h_2(x, y))$. Prove that

$$(\frac{\partial g}{\partial x}(x, y))^2 + (\frac{\partial g}{\partial y}(x, y))^2 = (\frac{\partial f}{\partial u}(u, v))^2 + (\frac{\partial f}{\partial v}(u, v))^2.$$

**Solution.** Handwritten.

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