Fachbereich Mathematik
Dr. L. Leuştean
E. Briseid, S. Herrmann

TECHNISCHE UNIVERSITÄT DARMSTADT

06.07.2006

12. Home work Analysis II for MCS
Summer Term 2006

(H12.1)

Compute the derivative of the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(x, y, z) = (x \sin(y) \cos(z), x \sin(y) \sin(z), x \cos(y)).$$

Solution.

We have

$$J_f(x, y, z) = \begin{pmatrix} \sin(y)\cos(z) & x\cos(y)\cos(z) & -x\sin(y)\sin(z) \\ \sin(y)\sin(z) & x\cos(y)\sin(z) & x\sin(y)\cos(z) \\ \cos(y) & -x\sin(y) & 0 \end{pmatrix}.$$

Since all the partial derivatives are continuous it follows that f is differentiable and that

$$f'(x, y, z)(h) = J_f(x, y, z) \cdot h,$$

for all $h \in \mathbb{R}^3$. ■

(H12.2) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0). \end{cases}$$

Prove the following assertions.

(i) The partial derivatives $\partial_j f(0, 0)$ exist for $j = 1, 2$.

(ii) The directional derivative $D_v f(0, 0)$ does not exist if v is not a multiple of the standard unit vectors e_1, e_2.

(iii) The function f is not differentiable.

Solution.

(i) We have

$$\partial_1 f(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

and

$$\partial_2 f(0, 0) = \lim_{h\to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

(ii) Let $v = (v_1, v_2)$ be not a multiple of a standard unit vector, i.e. $v_1 \cdot v_2 \neq 0$. Then

$$\frac{f((0, 0) + hv) - f(0, 0)}{h} = \frac{1}{h} \cdot \frac{hv_1 \cdot hv_2}{h^2 v_1^2 + h^2 v_2^2} = \frac{1}{h} \cdot \frac{v_1 v_2}{v_1^2 + v_2^2},$$

and therefore

$$\lim_{h\to 0} \frac{f((0, 0) + hv) - f(0, 0)}{h}$$

does not exist, since $v_1 \cdot v_2 \neq 0$.

(iii) Since it is not the case that every directional derivative exists in $(0, 0)$, it follows by Proposition 9.21 that f is not differentiable in $(0, 0)$. ■

(H12.3)

Let $n \in \mathbb{N}$, and let $D \subseteq \mathbb{R}^n$ be open. Let $f : D \to \mathbb{R}$ be differentiable at $a \in \mathbb{R}^n$. Let $v \in \mathbb{R}^n$,

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Prove that

$$D_v f(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) \cdot v_i.$$

Solution.

By Propositions 9.21 and 9.28,

$$D_v f(a) = f'(a)(v) = J_f(a) \cdot v.$$

We have

$$J_f(a) = \left(\frac{\partial f}{\partial x_1}(a) \ \dots \ \frac{\partial f}{\partial x_n}(a) \right).$$

Thus

$$J_f(a) \cdot v = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) \cdot v_i.$$

■