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11. Home work Analysis II for MCS
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(H11.1)

Compute the arc length of the following curves.

(i) Let $0 < a < b < \infty$. We define $f : [a,b] \to \mathbb{R}^2$, $f(t) = (t^3, \frac{3}{2}t^2)$.

(ii) We define

$$\gamma : [0, 2\pi] \to \mathbb{R}^2, \qquad t \mapsto ((1+\cos t)\cos t, (1+\cos t)\sin t).$$

Solution.

(i) We have $f'(t) = (3t^2, 3t)$ and $\|f'(t)\|_2^2 = 9t^4 + 9t^2$. Hence by Theorem 8.21,

$$\begin{aligned} L(f) &= \int_a^b 3\sqrt{t^4+t^2}\mathrm{d}t = \int_{a^2}^{b^2} 3\sqrt{u^2+u}\cdot\frac{1}{2\sqrt{u}}\mathrm{d}u = \int_{a^2}^{b^2} \frac{3}{2}\sqrt{u+1}\mathrm{d}u \\ &= [(u+1)^{\frac{3}{2}}]_{a^2}^{b^2} = (b^2+1)^{\frac{3}{2}} - (a^2+1)^{\frac{3}{2}}. \end{aligned}$$

(ii) We write $r(t) = 1 + \cos t$. Then

$$\gamma'(t) = (r'(t)\cos t - r(t)\sin t, r'(t)\sin t + r(t)\cos t),$$

so

$$\|\gamma'(t)\|_2^2 = \|(r'(t)\cos t - r(t)\sin t, r'(t)\sin t + r(t)\cos t)\|_2^2,$$

that is,

$$\|\gamma'(t)\|_2^2 = r'(t)^2 + r(t)^2.$$

Now $r'(t)^2 + r(t)^2 = 1 + 2\cos t + \cos^2 t + \sin^2 t = 2(1+\cos t)$, so by Theorem 8.21 we have

$$L(\gamma) = \int_0^{2\pi} \sqrt{r(t)^2 + r'(t)^2}\mathrm{d}t.$$

And

$$\begin{aligned} \int_0^{2\pi} \sqrt{r(t)^2+r'(t)^2}\mathrm{d}t &= \sqrt{2}\int_0^{2\pi}\sqrt{1+\cos t}\mathrm{d}t \\ &= \sqrt{2}\int_0^{\pi}\sqrt{1+\cos t}\mathrm{d}t + \sqrt{2}\int_{\pi}^{2\pi}\sqrt{1+\cos t}\mathrm{d}t \\ &= \sqrt{2}\int_0^{\pi}\sqrt{1+\cos t}\mathrm{d}t + \sqrt{2}\int_0^{\pi}\sqrt{1+\cos s}\mathrm{d}s \quad \text{subst. } s = t - 2\pi \\ &= 2\sqrt{2}\int_0^{\pi}\sqrt{1+\cos t}\mathrm{d}t. \end{aligned}$$

Substituting $u = \cos t$ gives

$$2\sqrt{2}\int_0^{\pi}\sqrt{1+\cos t}\mathrm{d}t = 2\sqrt{2}\int_{-1}^{1}\frac{\sqrt{1+u}}{\sqrt{1-u^2}}\mathrm{d}u.$$

Using $\sqrt{1+u}\cdot\sqrt{1-u} = \sqrt{1-u^2}$, we obtain

$$\int_{-1}^{1}\frac{\sqrt{1+u}}{\sqrt{1-u^2}}\mathrm{d}u = [-2\sqrt{1-u}]_{-1}^{1}.$$

So $2\sqrt{2}\cdot[-2\sqrt{1-u}]_{-1}^{1} = 8$ gives $L(\gamma) = 8$. ■

(H11.2)

We define the following relation for paths. Two paths $f_j : [a_j, b_j] \to X$, $j = 1, 2$, are called *equivalent* if there is a strictly isotone surjective function $\sigma : [a_1, b_1] \to [a_2, b_2]$ such that $f_1 = f_2 \circ \sigma$. That is, if there exists a change of parameters $\sigma : [a_1, b_1] \to [a_2, b_2]$ such that f_1 is obtained from f_2 by σ, and moreover, σ is strictly isotone.

Prove that this relation is indeed an equivalence relation on the set of all curves in a fixed metric space X.

(This is Remark 8.15 in the handouts.)

Solution. First note that for strictly isotone surjective functions $\sigma_1 : [a_1, b_1] \to [a_2, b_2]$ and $\sigma_2 : [a_2, b_2] \to [a_3, b_3]$ respectively, we have that σ_1 is bijective and σ_1^{-1} is also strictly isotone and surjective. Furthermore, $\sigma_2 \circ \sigma_1$ is strictly isotone and surjective.

For $x \neq y \in [a_1, b_1]$ we either have $x < y$ or $y < x$. Then since σ_1 is strictly isotone we get either $\sigma_1(x) < \sigma_1(y)$ or $\sigma_1(y) < \sigma_1(x)$. Therefore $\sigma_1(y) \neq \sigma_1(x)$. Hence σ_1 is bijective and σ_1^{-1} exists.

For $s < t \in [a_2, b_2]$ we get $x, y \in [a_1, b_1]$ with $\sigma_1(x) = s < t = \sigma_1(y)$. If $x > y$ then $\sigma_1(x) > \sigma_1(y)$ yields a contradiction. Therefore we have that $\sigma_1^{-1}(s) = x < y = \sigma_1^{-1}(t)$.

Furthermore, by

$$x < y \quad\Rightarrow\quad \sigma_1(x) < \sigma_1(y) \quad\Rightarrow\quad \sigma_2(\sigma_1(x)) < \sigma_2(\sigma_1(y))$$

and since for $t \in [a_3, b_3]$

$$x = \sigma_1^{-1} \circ \sigma_2^{-1}(t) \quad \Rightarrow \quad \sigma_2 \circ \sigma_1(x) = t,$$

we get that $\sigma_2 \circ \sigma_1$ is strictly isotone and surjective.

Now we are ready to solve the exercise:

Reflexivity: Set $\sigma = id$ to get $f = f \circ id$.

Symmetry: If $f_1 = f_2 \circ \sigma$, then we get by

$$f_1(\sigma^{-1}(t)) = f_2(\sigma(\sigma^{-1}(t))) = f_2(t)$$

that the relation is symmetric.

Transitivity: Let $f_1 = f_2 \circ \sigma_1$ and $f_2 = f_3 \circ \sigma_2$. Then with $\sigma = \sigma_2 \circ \sigma_1$ we get $f_1 = f_3 \circ \sigma_2 \circ \sigma_1 = f_3 \circ (\sigma_2 \circ \sigma_1) = f_3 \circ \sigma$. ∎

(H11.3)

Prove the following:

The image of a rectifiable curve in $\mathbb{R}^2$ does not contain the square $[0,1]^2$.

(Conclude that γ_{sc} as defined in Tutorial 11 is not rectifiable.)

Hint: Let $\delta : [a,b] \to \mathbb{R}^2$ be a rectifiable curve and assume that $Q = [0,1]^2 \subseteq \delta([a,b])$. Let $n \in \mathbb{N}$ and define a subset $M \subseteq Q$ by

$$M = \left\{ \left(\frac{p}{n}, \frac{q}{n} \right) : 0 \leq p, q \leq n \right\}.$$

Remark that according to our assumption there are points $t_1 < t_2 < \ldots < t_{(n+1)^2}$ of $[a,b]$ such that $\delta(\{t_1, \ldots, t_{(n+1)^2}\}) = M$. Consider a partition P of $[a,b]$ containing the points $t_1 < \ldots < t_{(n+1)^2}$.

Solution. Let $\delta : [a,b] \to \mathbb{R}^2$ be a rectifiable curve. Assume that $Q = [0,1]^2 \subseteq \delta([a,b])$. Let $n \in \mathbb{N}$. Define a subset $M \subseteq Q$ by

$$M = \left\{ \left(\frac{p}{n}, \frac{q}{n} \right) : 0 \leq p, q \leq n \right\}.$$

Hence M consists of $(n+1)^2$ points, and since $M \subseteq Q \subseteq \delta([a,b])$ there are points $t_1 < t_2 < \ldots < t_{(n+1)^2}$ of $[a,b]$ such that $\delta(\{t_1, \ldots, t_{(n+1)^2}\}) = M$. Remark that $\|\delta(t_j) - \delta(t_{j-1})\|_2 \geq \frac{1}{n}$, i.e. the distance with respect to the Euclidean norm between two different points from M is at least $\frac{1}{n}$. Let P be any partition of $[a,b]$ containing the points $t_1 < \ldots < t_{(n+1)^2}$. We get that

$$V_P(\delta) \geq \sum_{j=2}^{(n+1)^2} \|\delta(t_j) - \delta(t_{j-1})\|_2 \geq \sum_{j=2}^{(n+1)^2} \frac{1}{n} = \frac{(n+1)^2 - 1}{n} > n.$$

Hence, for every $n \in \mathbb{N}$ we can find a partition P of $[a,b]$ such that $V_P(\delta) > n$, which contradicts the fact that δ is rectifiable. ∎