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10. Home work Analysis II for MCS
Summer Term 2006

(H10.1) Solution.

(i) Since $0 \in P_n$ we have

$$\|f - p_G\|_\infty \leq \|f - 0\|_\infty = \|f\|_\infty.$$

Hence $\|p_G\|_\infty \leq \|p_G - f\|_\infty + \|f\|_\infty \leq 2\|f\|_\infty.$

(ii) Define $\varphi_f : P_n \to \mathbb{R}$ by $\varphi_f(p) := \|f-p\|_\infty.$

Then φ_f is continuous (as a composition of continuous functions). Let

$$K_{f,n} := \{p \in P_n : \|p\|_\infty \leq 2\|f\|_\infty\}.$$

Then $K_{f,n}$ considered as a subspace of $(P_n, \|\cdot\|_\infty)$ is closed and bounded.

Since P_n is finite dimensional we conclude by Theorems 6.42 and 6.44, and also (T10.1), (G9.2) and (H9.1), that $K_{f,n}$ is a compact subspace of $(P_n \|\cdot\|_\infty)$.

Then by Theorem 3.52 φ_f attains its minimum on $K_{f,n}$. Let $p_6 \in K_{f,n}$ be s.t.

$$\|f - p_6\|_\infty = \inf\{\|f-p\|_\infty : p \in K_{f,n}\}.$$

For all $p \in P_n$ with $p \notin K_{f,n}$ we have

$\|p\|_\infty > 2\|f\|_\infty$. So $\|f-p\|_\infty + \|f\|_\infty \geq \|p\|_\infty$ gives

$\|f-p\|_\infty > \|f\|_\infty$. Since $0 \in K_{f,n}$ and

$\|f - 0\|_\infty = \|f\|_\infty$ we see that

$$\|f-p_6\|_\infty \leq \|f\|_\infty. \text{ Thus}$$

$$\|f - p_6\|_\infty = \inf\{\|f-p\|_\infty : p \in P_n\}.$$

$\square$

(H10.2) Solution.

(i) We have

$$(\forall n \in \mathbb{N})(\forall x \in \mathbb{R})(|\sin(nx)| \leq 1).$$

So for every $\varepsilon > 0$ we have for all $n \in \mathbb{N}$ with $n \geq \frac{1}{\varepsilon}$ that

$(\forall x \in \mathbb{R})\ \left(|f_n(x)| \le \frac{1}{n} \le \varepsilon \right).$

So f_n converges uniformly on $\mathbb{R}$ to 0.

(ii) We have $f_n'(x) = \frac{1}{n} \cdot n \cdot \cos(nx) = \cos(nx)$.

Since $\cos(n\pi) = -1$ for n odd and $\cos(n\pi) = 1$ for n even, we conclude that $(f_n'(\pi))_n$ does not converge. Hence $(f_n')_n$ does not converge pointwise. $\square$

(H10.3) Solution.

If (f_n) converges uniformly to f, then for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ st.

$|f(x) - f_n(x)| < \frac{\varepsilon}{2}$ for every $n \ge N$ and for every $x \in X$. Then for $m, n \ge N$ we have

$$|f_m(x) - f_n(x)| = |f_m(x) - f(x) + f(x) - f_n(x)|$$
$$\le |f_m(x) - f(x)| + |f(x) - f_n(x)| < \varepsilon$$

for all $x \in X$. So (f_n) is uniformly Cauchy.

Suppose now that (f_n) is uniformly Cauchy.

Then $(f_n(x))_n$ is a Cauchy sequence in $\mathbb{R}$ for each $x \in X$, hence convergent for each $x \in X$. Let $f : X \to \mathbb{R}$ be defined by $f(x) := \lim_{n\to\infty} f_n(x)$, for each $x \in X$. Let $\varepsilon > 0$. Let then $N \in \mathbb{N}$ be st. $|f_m(x) - f_n(x)| < \frac{\varepsilon}{2}$ for all $x \in X$ and all $m, n \geq N$. Then, for every $n \geq N$ and for all $x \in X$, we have

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)|$$

$$\leq \frac{\varepsilon}{2} + |f_m(x) - f(x)|,$$

for all $m \geq N$. Let $x \in X$, and let $m \geq N$ be st. $|f_m(x) - f(x)| \leq \frac{\varepsilon}{2}$. Then we get

$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$, and this holds for all $x \in X$ and all $n \geq N$. So (f_n) converges uniformly to f on X.

$\square$