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9. Home work Analysis II for MCS
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(H9.1) Solution.

(i) Suppose $(V, \|\cdot\|_1)$ is a Banach space, and suppose $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(V, \|\cdot\|_2)$. We must show that $(x_n)_{n\in\mathbb{N}}$ is convergent in $(V, \|\cdot\|_2)$.

We know from (G9.2) (i) that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_1)$.

Thus $(x_n)_{n\in\mathbb{N}}$ is convergent in $(V, \|\cdot\|_1)$, say $\lim_{n\to\infty} x_n = x$. Then by (G9.2)(ii) $\lim_{n\to\infty} x_n = x$ also in $(V, \|\cdot\|_2)$. Thus every Cauchy sequence is convergent in $(V, \|\cdot\|_2)$, and so $(V, \|\cdot\|_2)$ is a Banach space.

The same argument applies to show that if $(V, \|\cdot\|_2)$ is a Banach space, then so is $(V, \|\cdot\|_1)$.

(ii) Assume A closed in $(V, \|\cdot\|_1)$. Then $V \setminus A$ is open in $(V, \|\cdot\|_1)$, and by (G9.2)(iii) we have that $V \setminus A$ is open in $(V, \|\cdot\|_2)$. Thus A is closed in $(V, \|\cdot\|_2)$.

Assume now A compact in $(V, \|\cdot\|_1)$, i.e. every sequence in A has a cluster point in A with respect to the metric induced by $\|\cdot\|_1$. (See (T12.2) from Analysis I) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A, and let $x \in A$ be a cluster point of $(x_n)_{n\in\mathbb{N}}$ with respect to the metric induced by $\|\cdot\|_1$. By (T11.2) from Analysis I $x \in A$ is a cluster point of $(x_n)_{n\in\mathbb{N}}$ with respect to the metric induced by $\|\cdot\|_1$ if and only if some subsequence of $(x_n)_{n\in\mathbb{N}}$

converges to x in $(V, \|\cdot\|_1)$. So let $(x_{n_k})_{k\in\mathbb{N}}$ be such a subsequence of $(x_n)_{n\in\mathbb{N}}$.

Then $\lim_{k\to\infty} x_{n_k} = x$ in $(V, \|\cdot\|_1)$, and so by (G9.2)(ii) also $\lim_{k\to\infty} x_{n_k} = x$ in $(V, \|\cdot\|_2)$.

Then $x \in A$ is a cluster point of $(x_n)_{n\in\mathbb{N}}$ with respect to the metric induced by $\|\cdot\|_2$, so A is compact in $(V, \|\cdot\|_2)$.

The same arguments give that if A is closed (resp. compact) in $(V, \|\cdot\|_2)$, then A is closed (resp. compact) in $(V, \|\cdot\|_1)$.

(iii) Assume A connected in $(V, \|\cdot\|_1)$.

Assume now that A is disconnected in $(V, \|\cdot\|_2)$.

Then there exist nonempty $U, W \subseteq A$ s.t. $U \cap W = \emptyset$ and $U \cup W = A$, and s.t. U and W are open in A with the metric induced by $\|\cdot\|_2$.

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We show that U and W are ~~open in~~ A also with the metric induced by $\|\cdot\|_1$. Then A would be disconnected in $(V, \|\cdot\|_1)$, contradicting the hypothesis. Thus we would have that A is indeed connected in $(V, \|\cdot\|_2)$.

That U is open in $\underbrace{A \text{ means that for}}_{\text{with the metric induced by } \|\cdot\|_2}$ each $u \in U$ there exist $\varepsilon > 0$ such that

$$\{x \in A : \|x - u\|_2 < \varepsilon\} \subseteq U.$$

We must show that for each $u \in U$ there exist $\varepsilon > 0$ such that

$$\{x \in A : \|x - u\|_1 < \varepsilon\} \subseteq U.$$

Since $\|\cdot\|_1$ and $\|\cdot|_2$ are equivalent we find $C \in]0, \infty[$ s.t.

$$(\forall x \in V)(\|x\|_2 \leq C\|x\|_1).$$

Let $u \in U$ and let $\varepsilon > 0$ be such that

$$\{x \in A : \|x - u\|_2 < \varepsilon\} \subseteq U.$$

Then since $\|x - u\|_1 < \frac{\varepsilon}{C}$ gives $\|x - u\|_2 < \varepsilon$ we get

$$\left\{x \in A : \|x - u\|_1 < \frac{\varepsilon}{C}\right\} \subseteq \{x \in A : \|x - u\|_2 < \varepsilon\} \subseteq U.$$

Thus U is open in A with the metric induced by $\|\cdot\|_1$. The same holds for W.

This is what we needed to show.

Notice that this argument which relays on Theorem 3.4 offers an alternative strategy for proving for instance (G9.2)(iii).

$\square$

(H9.2) Solution.

Let $(x_n)_{n\in\mathbb{N}}$ be Cauchy in V, and let $(\alpha_n)_{n\in\mathbb{N}}$ be convergent in $\mathbb{K}$. Let $\lim_{n\to\infty}\alpha_n = \alpha$.

Let $N\in\mathbb{N}$ be such that

$$(\forall m,n \geq N)(\|x_m - x_n\| < 1),$$

so for all $n\geq N$ we have $\|x_n\| < \|x_N\| + 1$.

Let $N'\in\mathbb{N}$ be s.t.

$$(\forall n \geq N')(|\alpha - \alpha_n| < 1),$$

so for all $n \geq N'$ we have $|\alpha_n| < |\alpha| + 1$.

Let $\varepsilon > 0$.

We now let $N_1 \geq \max\{N, N'\}$ be such that

$$(\forall m, n \geq N_1)\left(\|x_m - x_n\| < \frac{\varepsilon}{2(\max\{\|x_N\|, |\alpha|\}+1)}\right)$$

and

$$(\forall m, n \geq N_1)\left(|\alpha_n - \alpha_m| < \frac{\varepsilon}{2(\max\{\|x_N\|, |\alpha|\}+1)}\right).$$

Then for $m, n \geq N_1$ we have

$$\|\alpha_n x_n - \alpha_m x_m\| = \|(\alpha_n x_n - \alpha_m x_n) + (\alpha_m x_n - \alpha_m x_m)\|$$

$$\leq \|\alpha_n x_n - \alpha_m x_n\| + \|\alpha_m x_n - \alpha_m x_m\|$$

$$= \|x_n\| |\alpha_n - \alpha_m| + |\alpha_m| \|x_n - x_m\|$$

$$< (\|x_N\|+1)|\alpha_n - \alpha_m| + (|\alpha|+1)\|x_n - x_m\|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $(\alpha_n x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence.

$\square$

(H9.3) Solution.

Since
$$\{\|T(x)\| : x \in V, \|x\| = 1\} \subseteq \{\|T(x)\| : x \in V, \|x\| \leq 1\},$$
we get
$$\sup\{\|T(x)\| : x \in V, \|x\| = 1\} \leq \|T\|.$$
Let now $x \in V$ be such that $\|x\| \leq 1$, and such that $\|x\| \neq 0$. Then $\left\|\frac{x}{\|x\|}\right\| = 1$

and
$$\left\|T\left(\frac{x}{\|x\|}\right)\right\| = \frac{1}{\|x\|}\|T(x)\|.$$
Since $\frac{1}{\|x\|} \geq 1$ we get $\left\|T\left(\frac{x}{\|x\|}\right)\right\| \geq \|T(x)\|$.

Thus
$$\sup\{\|T(x)\| : x \in V, \|x\| \leq 1\} \leq \sup\{\|T(x)\| : x \in V, \|x\| = 1\}.$$
So
$$\|T\| = \sup\{\|T(x)\| : x \in V, \|x\| = 1\}.$$
$\square$