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8. Home work Analysis II for MCS
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(H8.1) Solution.

(i) Let $\lambda \in \mathbb{R}$, $\lambda > 1$. We write $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Then

$$\|\lambda x\| = (\lambda x_1)^2 + (\lambda x_2)^2 + (\lambda x_3)^2$$

$$= \lambda^2 x_1^2 + \lambda^2 x_2^2 + \lambda^2 x_3^2$$

$$= \lambda^2 \|x\|.$$

Since $|\lambda| < \lambda^2$ we conclude that N2 does not hold.

(ii) Consider $x = (0, 1, 0) \in \mathbb{R}^3$. Then $\|x\| = 0$ but $x \neq 0$. So N1 is violated.

(iii) We let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Then $\|x\| \geq 0$, since $|x_1|, |x_2|, |x_3| \geq 0$, and thus $\max\{|x_1|, |x_2|\} + |x_3| \geq 0$.

If $x=0$ then $|x_1|, |x_2|, |x_3| = 0$, and so $\|x\| = 0$. If on the other hand $\|x\| = 0$, then $\max\{|x_1|, |x_2|\} = 0$ and

$|x_3| = 0$, and thus $x_1, x_2, x_3 = 0$. So N1 is fulfilled. Let now $\lambda \in \mathbb{R}$.

Then

$$\|\lambda x\| = \max\{|\lambda x_1|, |\lambda x_2|\} + |\lambda x_3|$$

$$= \max\{|\lambda|\,|x_1|, |\lambda|\,|x_2|\} + |\lambda|\,|x_3|$$

$$= |\lambda| (\max\{|x_1|, |x_2|\} + |x_3|)$$

$$= |\lambda|\,\|x\|,$$

so N2 holds.

Let $x, y \in \mathbb{R}^3$. Then

$$\|x+y\| = \max\{|x_1+y_1|, |x_2+y_2|\} + |x_3+y_3|$$

$$\leq \max\{|x_1| + |y_1|, |x_2| + |y_2|\} + (|x_3| + |y_3|)$$

$$\leq \max\{|x_1|, |x_2|\} + \max\{|y_1|, |y_2|\} + (|x_3| + |y_3|)$$

$= \max\{|x_1|, |x_2|\} + |x_3| + \max\{|y_1|, |y_2|\} + |y_3|$

$= \|x\| + \|y\|.$

Thus N3 holds, and $\|\cdot\|$ is a norm.

(iv) Let $x = (1, 0, 0) \in \mathbb{R}^3$, $y = (0, 1, 0) \in \mathbb{R}^3$.

Then $\|x\| = 1$ and $\|y\| = 1$.

But $x + y = (1, 1, 0)$, so

$\|x+y\| = \left(\sqrt{|1|} + \sqrt{|1|} + \sqrt{|0|}\right)^2 = (1+1)^2 = 4.$

Thus $\|x+y\| > \|x\| + \|y\|$, and N3 does not hold.

$\square$

(H8.2).

We must show that $\|\cdot\| : V \times W \to \mathbb{R}$ satisfies N1, N2, N3 from definition 6.1 in the handouts. Let $(v, w) \in V \times W$.

Then $\|(v,w)\| = \|v\|_1 + \|w\|_2 \geq 0$, since $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms. So

$$\|(v,w)\| = 0 \iff (\|v\|_1 = 0 \text{ and } \|w\|_2 = 0) \iff$$
$$(v = 0 \text{ and } w = 0) \iff (v, w) = 0.$$

The last equivalence follows since we have $r(0,0) = (r0, r0) = (0,0)$ for all $r \in \mathbb{K}$, and since therefore $(0,0)$ is the zero vector in $V \times W$.

Let now $\lambda \in \mathbb{K}$. Then

$$\|\lambda(v,w)\| = \|(\lambda v, \lambda w)\| = \|\lambda v\|_1 + \|\lambda w\|_2$$
$$= |\lambda| \|v\|_1 + |\lambda| \|w\|_2 = |\lambda| \|(v,w)\|.$$

So N2 is satisfied. Let now

$(v_1, w_1), (v_2, w_2) \in V \times W$. Then

$$\|(v_1, w_1) + (v_2, w_2)\| = \|(v_1 + v_2, w_1 + w_2)\|$$

$$= \|v_1 + v_2\|_1 + \|w_1 + w_2\|_2$$

$$\leq (\|v_1\|_1 + \|v_2\|_1) + (\|w_1\|_2 + \|w_2\|_2)$$

$$= (\|v_1\|_1 + \|w_1\|_2) + (\|v_2\|_1 + \|w_2\|_2)$$

$$= \|(v_1, w_1)\| + \|(v_2, w_2)\|.$$

So N3 holds, and $\|\cdot\|$ is a norm on $V \times W$.

$\square$

(H8.3) Let $v \in V$ with $v \neq 0$. Then $\|v\| > 0$, and $\frac{r \cdot v}{\|v\|} \in V$. We have

$$\left\|\frac{r}{\|v\|} \cdot v\right\| = \left|\frac{r}{\|v\|}\right| \cdot \|v\| = \frac{r}{\|v\|} \cdot \|v\| = r.$$

Suppose $M \in \mathbb{R}$ satisfies $M > d(x,y)$ for all $x, y \in V$ with $d : V \times V \to \mathbb{R}$ the metric induced by $\|\cdot\|$, i.e. $d(x,y) = \|x-y\|$.
Let $r > M$.
We have shown that there exists $x \in V$ with $\|x\| = r$. Then

$$d(x,0) = \|x-0\| = \|x\| = r > M,$$

a contradiction. Thus V is unbounded. $\square$