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7. Homework Analysis II for MCS
SS 2006

(H7.3) Solution.

We use the Hölder inequality from (T6.1) and get

$$\int_a^b |fg| \leq \left(\int_a^b |f|^2\right)^{\frac{1}{2}} \left(\int_a^b |g|^2\right)^{\frac{1}{2}}$$

$$= \left(\int_a^b f^2\right)^{\frac{1}{2}} \left(\int_a^b g^2\right)^{\frac{1}{2}},$$

i.e.

$$\left(\int_a^b |fg|\right)^2 \leq \left(\int_a^b f^2\right)\left(\int_a^b g^2\right).$$

Now fg is continuous, and hence integrable. Furthermore, $|fg| = (fg)^+ + (fg)^-$ and

$$fg = (fg)^+ - (fg)^-,$$

So

$$\left|\int_a^b fg\right| = \left|\int_a^b (fg)^+ - \int_a^b (fg)^-\right|$$

and

$$\int_a^b |fg| = \int_a^b (fg)^+ + \int_a^b (fg)^-.$$

Since $(fg)^+, (fg)^- \geq 0$ we have

$$\int_a^b (fg)^+ + \int_a^b (fg)^- \geq \left|\int_a^b (fg)^+ - \int_a^b (fg)^-\right|,$$

and hence

$$\left|\int_a^b fg\right| \leq \int_a^b |fg|.$$

So

$$\left(\int_a^b fg\right)^2 \leq \left(\int_a^b |fg|\right)^2 \leq \left(\int_a^b f^2\right)\left(\int_a^b g^2\right).$$