Fachbereich Mathematik
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7. Home work Analysis II for MCS
Summer Term 2006

(H7.1)

Compute the following integrals.

(i)
$$\int_0^{2\pi} |\sin x| \, \mathrm{d}x.$$

(ii)
$$\int_2^{e^2} \frac{1}{x \log x} \, \mathrm{d}x.$$

(iii)
$$\int_0^1 \arctan x \, \mathrm{d}x.$$

Solution.

(i) Since $\sin(\pi + x) = -\sin x$ and we furthermore have $\sin(x) \geq 0$ for $x \in [0, \pi]$, we get
$$\int_\pi^{2\pi} |\sin x| \, \mathrm{d}x = \int_0^\pi |\sin(x+\pi)| \, \mathrm{d}x = \int_0^\pi |\sin x| \, \mathrm{d}x = \int_0^\pi \sin x \, \mathrm{d}x.$$
And hence
$$\int_0^{2\pi} |\sin x| \, \mathrm{d}x = \int_0^\pi |\sin x| \, \mathrm{d}x + \int_\pi^{2\pi} |\sin x| \, \mathrm{d}x = 2\int_0^\pi \sin x \, \mathrm{d}x.$$
So
$$\int_0^{2\pi} |\sin x| \, \mathrm{d}x = 2(-\cos \pi + \cos 0) = 4.$$

(ii) We substitute $u(x) = \log x$ and get
$$\int_2^{e^2} \frac{1}{x \log x} \, \mathrm{d}x = \int_{\log 2}^2 \frac{1}{u} \, \mathrm{d}u = \log u|_{\log 2}^2 = \log(2) - \log\log(2).$$

(iii) Since
$$(\arctan x)' = \frac{1}{1+x^2},$$
we get by integration by parts that
$$\int_0^1 \arctan x \, \mathrm{d}x = x \arctan x|_0^1 - \int_0^1 \frac{x}{1+x^2} \, \mathrm{d}x = \arctan 1 - \frac{1}{2}\log 2 = \frac{\pi}{4} - \frac{1}{2}\log 2.$$
For the computation of $\int_0^1 \frac{x}{1+x^2} \, \mathrm{d}x$, see example 2 in the section of the lectures dealing with substitution.

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(H7.2)

Compute the following integrals.

(i)
$$\int_1^e \frac{\log x}{x} \, \mathrm{d}x.$$

(ii)
$$\int_0^\pi x^2 \cos x \, \mathrm{d}x.$$

(iii)
$$\int_0^1 x \arctan x \, \mathrm{d}x.$$

Solution.

(i) We use partial integration
$$\int_1^e \frac{\log x}{x} \, \mathrm{d}x = (\log x)^2|_1^e - \int_1^e \frac{\log x}{x} \, \mathrm{d}x.$$
Therefore we have
$$\int_1^e \frac{\log x}{x} \, \mathrm{d}x = \frac{1}{2}(\log x)^2|_1^e = \frac{1}{2}.$$

(ii) We have
$$\int_0^\pi x^2 \cos x \, \mathrm{d}x = \int_0^\pi x^2 \sin' x \, \mathrm{d}x = x^2 \sin x|_0^\pi - 2\int_0^\pi x \sin x \, \mathrm{d}x = x^2 \sin x|_0^\pi + 2\int_0^\pi x \cos' x \, \mathrm{d}x,$$
which again gives
$$\int_0^\pi x^2 \cos x \, \mathrm{d}x = (x^2 \sin x + 2x \cos x)|_0^\pi - 2\int_0^\pi \cos x \, \mathrm{d}x = -2\pi.$$

(iii) Integration by parts yields

$$\int_0^1 x \arctan x \, \mathrm{d}x = \frac{1}{2} x^2 \arctan x |_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} \, \mathrm{d}x.$$

We notice that

$$\int_0^1 \frac{x^2}{1+x^2} \, \mathrm{d}x = \int_0^1 \frac{(1+x^2)-1}{1+x^2} \, \mathrm{d}x = \int_0^1 \left(1 - \frac{1}{1+x^2}\right) \mathrm{d}x.$$

And

$$\int_0^1 \left(1 - \frac{1}{1+x^2}\right) \mathrm{d}x = (x - \arctan x)|_0^1,$$

so

$$\int_0^1 x \arctan x \, \mathrm{d}x = (\frac{1}{2} x^2 \arctan x - \frac{1}{2} x + \frac{1}{2} \arctan x)|_0^1 = \frac{\pi}{4} - \frac{1}{2}.$$

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(H7.3)

Prove the **Cauchy-Schwarz-Bunyakovski Inequality** for continuous $f, g : [a, b] \to \mathbb{R}$, i.e.

$$(\int_a^b fg)^2 \leq (\int_a^b f^2)(\int_a^b g^2),$$

for $a < b \in \mathbb{R}$.

Solution. Handwritten.

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