Fachbereich Mathematik
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6. Home work Analysis II for MCS
Summer Term 2006

(H6.1)

(i) Compute the following integrals using the Fundamental Theorem of Differential and Integral Calculus:

(a) $\int_1^e \frac{1}{x}\,dx$.

(b) $\int_0^\pi (x^n + \cos x)\,dx$.

(c) $\int_0^{\frac{\pi}{4}} \frac{1}{1+x^2}\,dx$.

(ii) Prove that

$$1 \le \int_0^1 \exp(x^2)\,dx \le e.$$

Solution.

(i) (a)

$$\int_1^e \frac{1}{x}\,dx = \ln x|_1^e = \ln e - \ln 1 = 1 - 0 = 1.$$

(b)

$$\begin{aligned}\int_0^\pi (x^n + \cos x)\,dx &= \int_0^\pi x^n\,dx + \int_0^\pi \cos x\,dx = \left.\frac{x^{n+1}}{n+1}\right|_0^\pi + \sin x|_0^\pi \\ &= \frac{\pi^{n+1}}{n+1} + (\sin\pi - \sin 0) = \frac{\pi^{n+1}}{n+1}.\end{aligned}$$

(c)

$$\int_0^{\frac{\pi}{4}} \frac{1}{1+x^2}\,dx = \arctan x|_0^{\frac{\pi}{4}} = \arctan\frac{\pi}{4} - \arctan 0 = \arctan\frac{\pi}{4} - 0 = \arctan\frac{\pi}{4}.$$

(ii) The function $f : [0,1] \to \mathbb{R}$, $f(x) = \exp(x^2)$ is isotone, so applying (G6.2)(ii), we get that

$$f(0) \le \int_0^1 f(x)\,dx \le f(1),$$

that is,

$$1 \le \int_0^1 \exp(x^2)\,dx \le e.$$

■

(H6.2)

Let $f : [0,1] \to [0,\infty[$ be a continuous function and assume that $\int_0^1 f(x)dx = 0$. Show that $f(x) = 0$ for all $x \in [0,1]$.

Solution.

Assume that $f(x_0) > 0$ for some $x_0 \in [0,1]$. Then $f(x_0) = \varepsilon$ for some $\varepsilon > 0$. Since f is continuous there exists a $\delta > 0$ with $\delta \le 1$ such that

$$(\forall x \in [0,1])(|x - x_0| \le \delta \Rightarrow f(x) \ge \varepsilon/2).$$

Hence

$$\int_0^1 f(x)dx \ge \delta \cdot \varepsilon/2 > 0.$$

Which contradicts the hypothesis. ■

(H6.3)

Let $f : [a,b] \to \mathbb{R}$ be such that f is of class C^2 and $f(a) = f(b)$. Prove that

$$\int_a^b x f''(x)\,dx = bf'(b) - af'(a).$$

Hint: Find an antiderivative of $xf''(x)$ and apply the Fundamental Theorem of Differential and Integral Calculus.

Solution.

Let us consider the function $F : [a,b] \to \mathbb{R}$, $F(x) = xf'(x) - f(x)$. Then F is of class C^1 and $F'(x) = xf''(x)$. Applying the Fundamental Theorem, we get that

$$\int_a^b x f''(x)\,dx = F(b) - F(a) = bf'(b) - af'(a).$$

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Orientation Colloquium

The Department of Mathematics' Research Groups present themselves.

Monday, 29.05.2006 – 16:15-17:15 – S207/109

Prof. Dr. Burkhard Kümmerer

FG Algebra, Geometrie und Funktionalanalysis

"Im Dreiländereck Funktionalanalysis – Stochastik – Mathematische Physik"

After the talk there will be a relaxed get-together (coffee, tea and biscuits) in S215/219, where interested people can discuss the talk and become more acquainted with the lecturer.