Fachbereich Mathematik
Dr. L. Leuştean
E. Briseid, S. Herrmann

TECHNISCHE UNIVERSITÄT DARMSTADT

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5. Home work Analysis II for MCS
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(H5.1)

Let $f, g : [a, b] \to \mathbb{R}$ be real functions. Prove the following statements:

(i) $|f| \geq 0$ and ($|f| = 0$ if and only if $f = 0$).

(ii) $|f + g| \leq |f| + |g|$ and $|fg| = |f| \cdot |g|$.

(iii) $|f| = f^+ + f^-$ and $f = f^+ - f^-$.

Solution.

(i) We have that $|f|(x) = |f(x)| \geq 0$, thus $|f| \geq 0$. If $|f|(x) = 0$ for all $x \in [a, b]$, then we get $|f(x)| = 0$ for all $x \in [a, b]$, hence $f(x) = 0$ for all $x \in [a, b]$ and therefore $f = 0$. The other direction is clear.

(ii) We have that $|f+g|(x) = |(f+g)(x)| = |f(x)+g(x)| \leq |f(x)|+|g(x)| = |f|(x)+|g|(x)$ for all $x \in [a, b]$. Hence $|f + g| \leq |f| + |g|$.

Moreover, $|fg|(x) = |(fg)(x)| = |f(x)g(x)| = |f|(x) \cdot |g|(x)$ for all $x \in [a, b]$. Hence $|fg| = |f| \cdot |g|$.

(iii) We remark that

$$|f|(x) = |f(x)| = \begin{cases} f(x), & \text{if } f(x) \geq 0 \\ -f(x), & \text{if } f(x) \leq 0 \end{cases} = f^+(x) + f^-(x).$$

Moreover we see that

$$f(x) = \begin{cases} f^+(x), & \text{if } f(x) \geq 0 \\ -f^-(x), & \text{if } f(x) \leq 0 \end{cases} = f^+(x) - f^-(x).$$

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(H5.2)

Let $a < b \in \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$, $f(x) = x^2$. Prove that f is integrable and compute

$$\int_a^b f dx.$$

Hint: Use (T5.1)(ii).

Solution.

Since f is continuous it is integrable. In order to compute its integral, we apply (T5.1)(ii). For every $n \in \mathbb{N}$, we define the partition $P_n = (a = x_0^{(n)} < x_1^{(n)} < \ldots < x_n^{(n)} = b)$, where $x_i^{(n)} = a + i \cdot \frac{b-a}{n}$ for all $0 \leq i \leq n$. Then $(P_n)_{n\in\mathbb{N}}$ is a sequence of partitions of $[a, b]$ satisfying

$$\lim_{n\to\infty} \|P_n\| = \lim_{n\to\infty} \frac{b-a}{n} = 0.$$

Consider now the intermediate points $\xi_i^{(n)} = x_i^{(n)}$, $1 \leq i \leq n$. Then

$$\begin{aligned}
\sigma(P_n, f, \xi^{(n)}) &= \sum_{i=1}^{n} f(\xi_i^{(n)})(x_i^{(n)} - x_{i-1}^{(n)}) = \sum_{i=1}^{n} f\left(a + i \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n} \\
&= \frac{b-a}{n} \sum_{i=1}^{n} \left(a^2 + 2ia\frac{b-a}{n} + i^2 \left(\frac{b-a}{n}\right)^2\right) \\
&= \frac{b-a}{n} \left(na^2 + \frac{2a(b-a)}{n} \sum_{i=1}^{n} i + \left(\frac{b-a}{n}\right)^2 \sum_{i=1}^{n} i^2\right) \\
&= \frac{b-a}{n} \left(na^2 + \frac{2a(b-a)}{n} \cdot \frac{n(n+1)}{2} + \frac{(b-a)^2}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right) \\
&= a^2(b-a) + a(b-a)^2\frac{n+1}{n} + (b-a)^3 \frac{(n+1)(2n+1)}{6n^2}
\end{aligned}$$

It follows that

$$\begin{aligned}
\lim_{n\to\infty} \sigma(P_n, f, \xi^{(n)}) &= a^2(b-a) + a(b-a)^2 + (b-a)^3\frac{1}{3} \\
&= a^2b - a^3 + ab^2 - 2a^2b + a^3 + (b^3 - 3ab^2 + 3a^2b - a^3)\frac{1}{3} = \frac{1}{3}(b^3 - a^3).
\end{aligned}$$

Using now (T5.1)(ii), we get that

$$\int_a^b f dx = \lim_{n\to\infty} \sigma(P_n, f, \xi^{(n)}) = \frac{1}{3}(b^3 - a^3).$$

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(H5.3)

Let $a < b \in \mathbb{R}$, and let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $]a, b[$. Assume that $f(a) \leq g(a)$ and that $f'(x) < g'(x)$ for $x \in]a, b[$. Show that $f(x) < g(x)$ for $x \in]a, b]$.

Solution.

Define $h : [a, b] \to \mathbb{R}$ by $h(x) := g(x) - f(x)$. Then $h(a) \geq 0$ and $h'(x) > 0$ for $x \in]a, b[$. Thus $h(x) > 0$ for $x \in]a, b]$. ■