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4. Homework Analysis II for MCS
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(H4.2) Solution

Define $h_1 : [0, \frac{1}{1+|a|}[\to \mathbb{R}$ by $h_1(x) := \begin{cases} f(\frac{1}{x}) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$

Define likewise $h_2 [0, \frac{1}{1+|a|}[\to \mathbb{R}$ by

$$h_2(x) := \begin{cases} g(\frac{1}{x}) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Then we have

(i) There is a $\delta > 0$ such that $h_2(x) \neq 0$ and $h_2'(x) \neq 0$ for all $x \in]0, \delta[$.

(ii) $\lim\limits_{\substack{x \to 0 \\ x > 0}} \frac{h_1'(x)}{h_2'(x)} = l$, since

$$\lim_{\substack{x \to 0 \\ x > 0}} \frac{h_1'(x)}{h_2'(x)} = \lim_{\substack{x \to 0 \\ x > 0}} \frac{f'(\frac{1}{x}) \cdot (-x^{-2})}{g'(\frac{1}{x}) \cdot (-x^{-2})} = \lim_{\substack{x \to 0 \\ x > 0}} \frac{f'(\frac{1}{x})}{g'(\frac{1}{x})} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)},$$

and the latter limit exists and equals l.

(iii) $h_1(0) = h_2(0) = 0$.

Furthermore, h_1 and h_2 are continuous on $[0, \delta]$ for some $\delta > 0$. So we can use the rule of Bernoulli and de l'Hôpital as found in the book (Hofmann, prop. 4.54) and conclude that

$$\lim_{\substack{x \to 0 \\ x > 0}} \frac{h_1(x)}{h_2(x)} = \ell,$$

i.e.

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \ell.$$

NB:
Note that prop. 4.54 requires f and g to be continuous in a.