Fachbereich Mathematik
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4. Home work Analysis II for MCS
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(H4.1)

Determine the local maxima and minima of the following function:

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) := \frac{x}{1+x^2}.$$

Solution. We have that

$$\begin{aligned} f'(x) &= \frac{1+x^2-x\cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}, \\ f''(x) &= \frac{-2x(1+x^2)^2-(1-x^2)\cdot 2(1+x^2)\cdot 2x}{(1+x^2)^4} = \frac{-2x(1+x^2)-4x(1-x^2)}{(1+x^2)^3} \\ &= \frac{2x^3-6x}{(1+x^2)^3}. \end{aligned}$$

Hence the zeros of f' are $x = \pm 1$, since

$$f'(x) = \frac{1-x^2}{(1+x^2)^2} = 0 \quad \Leftrightarrow \quad 1-x^2 = 0.$$

Because of

$$f''(-1) = \frac{-2+6}{8} > 0, \quad \text{and} \quad f''(1) = \frac{2-6}{8} < 0,$$

f has a local minimum at $x = -1$ and a local maximum at $x = 1$. Because f is everywhere differentiable there are no other local extrema. ■

(H4.2)

Prove the following version of the rule of Bernoulli and de l'Hôpital:

Let $a \in \mathbb{R}$ and let $f :]a, \infty[\to \mathbb{R}$ and $g :]a, \infty[\to \mathbb{R}$ be functions. Assume that:

(1) There is a $M \in \mathbb{R}$ such that f and g are differentiable on $]M, \infty[$, and such that $g(x) \neq 0$ and $g'(x) \neq 0$ for $x > M$.

(2) The limit $l = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists.

(3) $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$.

Then the limit

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

exists and coincides with l.

(This exercise has with hindsight been modified to make life easier for you. The same holds true for (T4.2).)

Solution. Handwritten. ■