Fachbereich Mathematik
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3. Home work Analysis II for MCS
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(H3.1) Let I be a real interval, let $n \in \mathbb{N}$ and let $f : I \to \mathbb{R}$ be a $(n+1)$-times differentiable function with $f^{(n+1)} = 0$. Show that f is a polynomial of degree not greater than n.

Solution. Let $a \in I$. We have by Taylor's formula that

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(u)}{(n+1)!}(x-a)^{n+1}$$

for some u located properly between a and x. The assertion is then immediate from our assumption $f^{(n+1)}(u) = 0$. ■

(H3.2) Compute the following limits:

(i) $\lim\limits_{\substack{x\to 1\\ x\neq 1}} \frac{x^x - x}{1 - x + \log x}$; (ii) $\lim\limits_{\substack{x\to 0\\ x\neq 0}} \frac{x - \sin x}{x \sin x}$.

Hint: Use the Rule of Bernoulli and de l'Hôpital.

Solution.

(i) We apply the Rule of Bernoulli and de l'Hôpital for the functions

$$f, g :]0, \infty[\to \mathbb{R}, \quad f(x) = x^x - x, \quad g(x) = 1 - x + \log x.$$

We get

$$\lim_{\substack{x\to 1\\ x\neq 1}} \frac{x^x - x}{1 - x + \log x} = \lim_{\substack{x\to 1\\ x\neq 1}} \frac{x^x(\log x + 1) - 1}{-1 + \frac{1}{x}}.$$

Since

$$\lim_{\substack{x\to 1\\ x\neq 1}} x^x(\log x + 1) - 1 = \lim_{\substack{x\to 1\\ x\neq 1}} -1 + \frac{1}{x} = 0,$$

we apply the Rule of Bernoulli and de l'Hôpital again, and get

$$\begin{aligned} \lim_{\substack{x\to 1\\ x\neq 1}} \frac{x^x - x}{1 - x + \log x} &= \lim_{\substack{x\to 1\\ x\neq 1}} \frac{x^x(\log x + 1) - 1}{-1 + \frac{1}{x}} \\ &= \lim_{\substack{x\to 1\\ x\neq 1}} \frac{x^x(\frac{1}{x} + (\log x + 1)^2)}{-\frac{1}{x^2}} = \frac{1(1+1)}{-1} = -2. \end{aligned}$$

(ii) Applying twice the Rule of Bernoulli and de l'Hôpital, we get

$$\lim_{\substack{x\to 0\\ x\neq 0}} \frac{x - \sin x}{x \sin x} = \lim_{\substack{x\to 0\\ x\neq 0}} \frac{1 - \cos x}{\sin x + x \cos x} = \lim_{\substack{x\to 0\\ x\neq 0}} \frac{\sin x}{\cos x + \cos x - x \sin x} = \frac{0}{2} = 0.$$

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