Fachbereich Mathematik
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2. Home work Analysis II for MCS
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(H2.1)

Let us consider the polynomial function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^n + ax + b$, where $a, b \in \mathbb{R}$, $n \in \mathbb{N}$. Prove the following:

(i) if n is even, then f has at most two zeros,

(ii) if n is odd, then f has at most three zeros.

Hint: Apply (T1.1).

Solution. Let us consider the equation $f'(x) = 0$, that is

$$nx^{n-1} = -a.$$

(i) If n is even, then $n-1$ is odd, so the above equation has exactly one solution, namely $x_1 = \sqrt[n-1]{\frac{-a}{n}}$. By (T1.1), we get that f has at most two zeros.

(ii) If n is odd, then $n-1$ is even, and the above equation has at most two zeros, namely $x_{1,2} = \pm \sqrt[n-1]{\frac{-a}{n}}$, when $a < 0$. Thus, by (T1.1), f has at most three zeros.

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(H2.2)

Let $a, b \in \left]0, \infty\right[$ with $a < b$, and let $n \in \mathbb{N}$ with $n \geq 2$. Prove that

$$na^{n-1}(b-a) < b^n - a^n < nb^{n-1}(b-a). \tag{1}$$

Hint: Use the Mean Value Theorem.

Solution. Let us consider the function $f : [a, b] \to \mathbb{R}, f(x) = x^n$. Then f is differentiable, and $f'(x) = nx^{n-1}$, so we can apply the Mean Value Theorem to get $c \in \left]a, b\right[$ such that

$$b^n - a^n = f(b) - f(a) = f'(c)(b-a) = nc^{n-1}(b-a).$$

Since $0 < a < c < b$ and $n - 1 \geq 1$ we get that $a^{n-1} < c^{n-1} < b^{n-1}$, so (1) follows immediately. ■