Fachbereich Mathematik
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1. Home work Analysis I for MCS
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(H1.1) Use Taylor's Theorem to estimate e with error less than 10^{-3}. Consider it known that $e < 3$.

Solution. We consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) := \exp(x)$. Then $f^{(n)}(x) = \exp(x)$ for all $n \in \mathbb{N}$. Taylor's Theorem implies that there exists $u \in]0,1[$ such that

$$\exp(1) = \exp(0) + \sum_{k=1}^{n-1} \frac{1}{k!}\exp(0)(1-0)^k + \frac{1}{n!}\exp(u)(1-0)^n = 1 + \sum_{k=1}^{n-1}\frac{1}{k!} + \frac{1}{n!}\exp(u).$$

Since exp is strictly monotone increasing on $\mathbb{R}$, we get

$$\left|\frac{1}{n!}\exp(u)\right| < \frac{e}{n!} < \frac{3}{n!}.$$

It is enough if we assure

$$\frac{3}{n!} \leq 10^{-3},$$

that is,

$$3000 \leq n!.$$

Thus we can let $n = 7$. Since

$$1 + \sum_{k=1}^{6}\frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} = \frac{1957}{720},$$

we have

$$\left|e - \frac{1957}{720}\right| < 10^{-3}.$$

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(H1.2) Use Taylor's Theorem to find the limit

$$\lim_{x\to 0}\frac{\cos x - 1 + x^2/2}{x^4}.$$

Solution. Taylor's Theorem implies that

$$\cos x = \cos 0 + \sum_{k=1}^{5}\frac{1}{k!}x^k\cos^{(k)}0 + \frac{1}{6!}x^6\cos^{(6)}u,$$

for some u located properly between 0 and x. That is,

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \frac{1}{6!}x^6\cos^{(6)}(u(x)),$$

for some $u : \mathbb{R} \to \mathbb{R}$ with $u(x)$ located properly between 0 and x. We write

$$R(x) := \frac{1}{6!}x^6\cos^{(6)}(u(x)),$$

and get

$$\lim_{x\to 0}\frac{\cos x - 1 + x^2/2}{x^4} = \lim_{x\to 0}\frac{1 - x^2/2 + x^4/24 + R(x) - 1 + x^2/2}{x^4}.$$

Since $\cos^{(6)}(u(x)) = -\cos(u(x))$ and $|\cos(u(x))| \leq 1$, we get

$$|R(x)| \leq \frac{x^6}{6!}.$$

Thus $\lim_{x\to 0}|R(x)/x^4| \leq \lim_{x\to 0} x^2/6! = 0$, and

$$\lim_{x\to 0}\frac{\cos x - 1 + x^2/2}{x^4} = \frac{1}{24}.$$

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