Fachbereich Mathematik
Dr. L. Leuştean
E. Briseid, S. Herrmann

TECHNISCHE UNIVERSITÄT DARMSTADT

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14. Exercise sheet Analysis II for MCS
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(G14.1)

(i) Show that the equation $x^3 + y^2 - 2xy = 0$ may be solved uniquely for (x, y) near $(1, 1)$ with respect to x and that the obtained function $x = \varphi(y)$ is continuously differentiable near $y = 1$. Calculate $\varphi'(1)$.

(ii) Show that φ is two times continuously differentiable near $y = 1$ and calculate $\varphi''(1)$.

(iii) Is the equation uniquely solvable with respect to y near $(1, 1)$?

Solution.

(i) Let

$$F : \mathbb{R}^2 \to \mathbb{R}, \quad F(x, y) = x^3 + y^2 - 2xy.$$

Then F is continuously differentiable with

$$\frac{\partial F}{\partial x}(x, y) = 3x^2 - 2y, \quad \frac{\partial F}{\partial y}(x, y) = 2y - 2x.$$

Since $F(1, 1) = 0$, and $\frac{\partial F}{\partial x}(1, 1) = 1 \neq 0$, we can solve the equation $F(x, y) = 0$ for (x, y) near $(a, b) = (1, 1)$ with respect to x using the Implicit Function Theorem. More precisely: there exists open neighborhoods U of $b = 1$, respectively V of $a = 1$ in $\mathbb{R}$, and a function $\varphi : U \to V$ such that $\varphi(1) = 1$, $F(\varphi(y), y) = F(a, b) = 0$ for all $y \in U$, and $\varphi(y)$ is the unique solution of the equation $F(x, y) = 0$ with $x \in V, y \in U$.

Calculation of $\varphi'(1)$:

By the Implicit Function Theorem, we have that φ is continuously differentiable on U, and for all $y \in U$

$$\varphi'(y) = -\frac{\frac{\partial F}{\partial y}(\varphi(y), y)}{\frac{\partial F}{\partial x}(\varphi(y), y)} = \frac{2\,y - 2\,\varphi(y)}{3\,\varphi(y)^2 - 2\,y} \tag{1}$$

In particular, $\varphi'(1) = -\frac{2-2}{3-2} = 0$.

(ii) Since φ is continuously differentiable on U, by (1) we get that φ' is also continuously differentiable on U. Thus, φ is two times continuously differentiable, and

$$\varphi''(y) = -\frac{2 - 2\varphi'(y)}{3\varphi(y)^2 - 2y} + \frac{2y - 2\varphi(y)}{(3\varphi(y)^2 - 2y)^2}(6\varphi(y)\varphi'(y) - 2).$$

In particular $\varphi''(1) = -2$.

(iii) As $\frac{\partial F}{\partial y}(1, 1) = 0$, the Implicit Function Theorem does not help to answer the question about the solvability of $F(x, y) = 0$ with respect to y for (x, y) near $(1, 1)$. However $F(x, y) = x^3 + y^2 - 2xy = 0$ is equivalent to

$$(y - x)^2 = x^2(1 - x). \tag{2}$$

Since the left hand side of (2) is never negative but the right hand side is negative for $x > 1$, we cannot solve $F(x, y) = 0$ for $x > 1$ with respect to y and there is no y with $F(x, y) = 0$. For $x = 1$ we obtain $(y - x)^2 = 0$, hence $y = x = 1$. For $x < 1$ we can explicitly solve the equation with respect to y and obtain two different solutions,

$$y_{1/2}(x) = x \pm x\sqrt{1 - x}\,.$$

■

(G14.2)

Prove that the map $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ with

$$F(x, y) = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$$

is locally invertible for $(x, y) \neq (0, 0)$. Is F also globally invertible? Compute the preimage $F^{-1}(\{(a, b)\})$ of an arbitrary point $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

Solution.

Obviously F is continously differentiable. To use the Inverse Function Theorem we have to prove that $F'(x, y)$ is invertible for all $(x, y) \neq 0$. Since

$$F'(x, y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix} \quad \text{hence} \quad \det(F'(x, y)) = 4(x^2 + y^2),$$

we get that $F'(x, y)$ is invertible for all $(x, y) \neq 0$.

The function F is not globally invertible because it is not injective: We have $F(x, y) = F(-x, -y)$.

Now we compute the preimage of an arbitrary point $(a, b) \in \mathbb{R}^2 \setminus \{0\}$: Let $(a, b) \in \mathbb{R}^2 \setminus \{0\}$. We search all $(x, y) \in \mathbb{R}^2$ with $F(x, y) = (a, b)$. We have

$$F(x, y) = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \implies 2xy = b.$$

We distinguish two cases:

(i) $b \neq 0$: Then $y \neq 0$, hence $x = \frac{b}{2y}$. If we put this in the equation $x^2 - y^2 = a$ we get

$$\begin{aligned}
\frac{b^2}{4y^2} - y^2 = a \quad &\overset{y\neq 0}{\Longleftrightarrow} \quad y^4 + ay^2 - \frac{b^2}{4} = 0 \\
&\Longrightarrow \quad y^2 = -\frac{a}{2} \pm \sqrt{\frac{a^2+b^2}{4}} \\
&\overset{y^2>0}{\Longrightarrow} \quad y^2 = -\frac{a}{2} + \sqrt{\frac{a^2+b^2}{4}} \\
&\Longrightarrow \quad y = \pm\sqrt{-\frac{a}{2} + \frac{\sqrt{a^2+b^2}}{2}}, \\
&\qquad\quad x = \pm\frac{b}{2\sqrt{-\frac{a}{2} + \frac{\sqrt{a^2+b^2}}{2}}}.
\end{aligned}$$

(ii) $b = 0$: Then we have $x = 0$ or $y = 0$ (and $a \neq 0$). If $a > 0$, we have $y = 0$ and $x = \pm\sqrt{a}$. If $a < 0$, we have $x = 0$ and $y = \pm\sqrt{a}$.

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(G14.3) (Supplementary)

Find the global maximum and minimum of the function

$$f(x, y) = 2x^2 + xy + \frac{5}{4}y^2 - 2x - 2y$$

on the unit square $S = [0, 1] \times [0, 1]$.

Hint: To compute the global extrema of a function f defined on a compact subset K of $\mathbb{R}^n$, you have to compute the local extrema on the interior of K and the global extrema on the boundary of K.

Solution. *Interior* $\mathring{S} = (0, 1) \times (0, 1)$*:*

The gradient is $\text{grad} f = (4x+y-2, x+\frac{5}{2}y-2)$. It is 0 at $x = \frac{1}{3}, y = \frac{2}{3}$. The point $\left(\frac{1}{3}, \frac{2}{3}\right)$ is inside the domain, so this is a candidate for global minimum or maximum. The value of the function is $f\left(\frac{1}{3}, \frac{2}{3}\right) = -1$.

Boundary: It is made of four line segments:

(i) $y = 0, 0 \leq x \leq 1 : f(x, 0) = 2x^2 - 2x$, critical point: $x = \frac{1}{2}$, possible candidates for minimum, maximum: $f(0, 0) = 0, f\left(\frac{1}{2}, 0\right) = -\frac{1}{2}, f(1, 0) = 0$.

(ii) $y = 1, 0 \leq x \leq 1 : f(x, 1) = 2x^2 - x - \frac{3}{4}$, critical point: $x = \frac{1}{4}$, possible candidates for minimum, maximum: $f(0, 1) = -\frac{3}{4}, f\left(\frac{1}{4}, 1\right) = -\frac{7}{8}, f(1, 1) = \frac{1}{4}$.

(iii) $x = 0, 0 \leq y \leq 1 : f(0, y) = \frac{5}{4}y^2 - 2y$, critical point: $y = \frac{4}{5}$, possible candidates for minimum, maximum: $f(0, 0) = 0, f\left(0, \frac{4}{5}\right) = -\frac{4}{5}, f(0, 1) = -\frac{3}{4}$.

(iv) $x = 1, 0 \leq y \leq 1 : f(1, y) = \frac{5}{4}y^2 - y$, critical point: $y = \frac{2}{5}$, possible candidates for minimum or maximum: $f(1, 0) = 0, f\left(1, \frac{2}{5}\right) = -\frac{1}{5}, f(1, 1) = \frac{1}{4}$.

So the global maximum is $f(1, 1) = \frac{1}{4}$, the global minimum is $f\left(\frac{1}{3}, \frac{2}{3}\right) = -1$. ■

(G14.4) (Supplementary)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ a continiously differentiable function and $f'(x)$ invertible for all $x \in \mathbb{R}^n$. Prove that f is open, i. e. $f(U)$ is open for each open set $U \subseteq \mathbb{R}^n$.

Solution. Let $y \in f(U)$ and $x \in U$ with $f(x) = y$. Since $f'(x)$ is invertible the Inverse Function Theorem yields neighbourhouds V of x and W of $f(x) = y$ such that $f|_V : V \to W$ has a continuously differentiable inverse function $g : W \to V$. So $f(V \cap U) = g^{-1}(V \cap U)$ is open because g is continuous. Hence, since $y \in f(V \cap U)$ there is a neighbourhood U_y of y with $U_y \subseteq f(V \cap U) \subseteq f(U)$. This holds for all $y \in f(U)$, so $f(U)$ is open. ■