Fachbereich Mathematik
Dr. L. Leuştean
E. Briseid, S. Herrmann

TECHNISCHE UNIVERSITÄT DARMSTADT

13.07.2006

13. Exercise sheet Analysis II for MCS
Summer Term 2006

(G13.1)

Let $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = e^z y + x^2y^2$ and $g : \mathbb{R} \to \mathbb{R}^3$,

$$g(t) = \begin{pmatrix} 2t^2 \\ \sin t \\ e^t \end{pmatrix}.$$

Compute the derivative of $f \circ g$ in two different ways.

(i) Directly by computing $h(t) = f(g(t))$ and differentiating h.

(ii) By using the chain rule.

Solution.

(i) We have

$$h(t) = e^{e^t} \sin t + 4t^4(\sin t)^2.$$

Hence

$$h'(t) = e^{e^t}(e^t \sin t + \cos t) + 16t^3(\sin t)^2 + 8t^4 \sin t \cos t.$$

(ii) The derivatives of f and g have the following associated matrices:

$$J_f(x, y, z) = \begin{pmatrix} 2xy^2 & e^z + 2x^2y & ye^z \end{pmatrix}$$

and

$$J_g(t) = \begin{pmatrix} 4t \\ \cos t \\ e^t \end{pmatrix}.$$

(The functions f and g are indeed differentiable, since the partial derivatives are all continuous.) Therefore the chain rule yields

$$J_h(t) = J_f(g(t)) \cdot J_g(t),$$

so

$$\begin{aligned} h'(t) &= 4t^2(\sin t)^2 4t + (e^{e^t} + 8t^4 \sin t)\cos t + \sin t(e^{e^t})e^t \\ &= e^{e^t}(e^t \sin t + \cos t) + 16t^3(\sin t)^2 + 8t^4 \sin t \cos t. \end{aligned}$$

■

(G13.2)

Let us consider the function

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x_1, x_2) := \begin{pmatrix} x_1 + x_2 \cos x_1 \\ x_2\, e^{x_1x_2} \end{pmatrix}.$$

Prove that the equation $f(x) = z$ for z near $(0, 0)$ possesses a unique solution $x = g(z)$ near $(0, 0)$. Show that g is continuously differentiable near $(0, 0)$ and compute $g'(0, 0)$.

Solution.

We have $f(0, 0) = (0, 0)$ and

$$J_f(x_1, x_2) = \begin{pmatrix} 1 - x_2 \sin x_1 & \cos x_1 \\ x_2^2\, e^{x_1x_2} & e^{x_1x_2} + x_1x_2\, e^{x_1x_2} \end{pmatrix},$$

in particular $J_f(0, 0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with $\det J_f(0, 0) = 1 \neq 0$. Since all the partial derivatives are continuous we can conclude that f is continuously differentiable. We can apply the Inverse Function Theorem to get an open neighborhood U of $(0, 0)$ and an open neighborhood V of $f(0, 0) = (0, 0)$ in $\mathbb{R}^2$, and a function $g : V \to U$ such that $f(g(t)) = t$ for all $t \in V$, and $g(f(x)) = x$ for all $x \in U$. That is, $f|_U$ is bijective and its inverse is g.

Thus, for any $z \in V$ the equation $f(x) = z$ has a unique solution in U, namely $x = g(z)$. Moreover, g is continuously differentiable, and

$$g'(0, 0) = (f'(0, 0))^{-1},$$

so the matrix associated with $g'(0, 0)$ is

$$J_g(0, 0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

■

(G13.3)

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$,

$$f(x, y, z) = \begin{pmatrix} z^2 + xy - 2 \\ z^2 + x^2 - y^2 - 1 \end{pmatrix}.$$

Remark that

$$f(1, 1, 1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consider the equations

$$\begin{aligned} z^2 + xy - 2 &= 0 \\ z^2 + x^2 - y^2 - 1 &= 0, \end{aligned}$$

and find for some neighborhood U of $z = 1$ in $\mathbb{R}$ a curve of solutions $\gamma : U \to \mathbb{R}^2$, $\gamma(z) = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$, passing through $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ at $z = 1$. Prove that γ is continuously differentiable and compute γ'.

Solution.

We have that

$$J_f(x, y, z) = \begin{pmatrix} y & x & 2z \\ 2x & -2y & 2z \end{pmatrix}.$$

Thus f is continuously differentiable, since all the partial derivatives are continuous. We also have

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix}.$$

At the point $(1, 1, 1)$ the determinant of this matrix is

$$\begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} = -4 \neq 0.$$

So we can apply the Implicit Function Theorem to get $\delta > 0$ and a function

$$\gamma :]1 - \delta, 1 + \delta[\to \mathbb{R}^2, \ \gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix},$$

such that $\gamma_1(1) = 1$, $\gamma_2(1) = 1$ and $f(\gamma_1(z), \gamma_2(z), z) = 0$ for all $z \in]1 - \delta, 1 + \delta[$. Moreover, γ is continuously differentiable, and writing $x = \gamma_1(z)$ and $y = \gamma_2(z)$ we have

$$\gamma'(z) = -\begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix}^{-1} \begin{pmatrix} \partial_3 f_1(x, y, z) \\ \partial_3 f_2(x, y, z) \end{pmatrix} = \begin{pmatrix} \frac{-z(x+2y)}{x^2+y^2} \\ \frac{z(y-2x)}{x^2+y^2} \end{pmatrix}.$$

■