Fachbereich Mathematik
Dr. L. Leuştean
E. Briseid, S. Herrmann

TECHNISCHE UNIVERSITÄT DARMSTADT

06.07.2006

12. Exercise sheet Analysis II for MCS
Summer Term 2006

(G12.1)

Let V and W be Banach spaces. We say that a function $A : V \to W$ is *affine* if there exist $c \in W$ and linear $T : V \to W$ such that

$$A(x) = T(x) + c$$

for all $x \in V$. Prove that any continuous affine function $A : V \to W$ is differentiable. Show also that if $A(x) = T(x) + c$ for $x \in V$, then $A'(x) = T$ for any $x \in V$.

Solution. Handwritten.

■

(G12.2)

Let
$$U = \{(r, \phi) : r > 0, \ \ 0 < \phi < 2\pi\} \subseteq \mathbb{R}^2,$$
and let
$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : U \to \mathbb{R}^2$$
be defined by
$$f_1(r, \phi) = r\cos\phi,$$
$$f_2(r, \phi) = r\sin\phi.$$
We write $x = f_1(r, \phi)$ and $y = f_2(r, \phi)$.

(i) Notice that f is injective with range
$$f(U) = \mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\}.$$
Show that f has a continuous inverse $f^{-1} : f(U) \to \mathbb{R}^2$.

(ii) Prove that f is differentiable, and compute the Jacobi matrix of f.

(iii) Prove that f^{-1} is differentiable, and compute the Jacobi matrix of f^{-1}.

Solution. Handwritten.

■

(G12.3)

Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$f(x, y) = \begin{cases} x^2 y \sin(\frac{y}{x}) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(i) Compute the partial derivatives of f where they exist and show that $\partial_1 f$ is not continuous in any point $(0, y)$ with $y \neq 0$.
Hint for the latter: For $y \neq 0$, consider the points $(x_n, y_n) = (y/(n\pi), y)$ for $n \in \mathbb{N}$.

(ii) Decide whether f is differentiable in $(0, y)$, for $y \in \mathbb{R}$.

Solution.

(i) For $x \neq 0$ we have
$$\partial_1 f(x, y) = 2xy\sin(\frac{y}{x}) - y^2\cos(\frac{y}{x})$$
and
$$\partial_2 f(x, y) = x^2\sin(\frac{y}{x}) + xy\cos(\frac{y}{x}).$$
For $x = 0$ we get
$$\partial_1 f(0, y) = \lim_{h \to 0} \frac{f(h, y) - f(0, y)}{h} = \lim_{h \to 0} \frac{h^2 y \sin(\frac{y}{h})}{h} = 0$$
and
$$\partial_2 f(0, y) = \lim_{h \to 0} \frac{f(0, y+h) - f(0, y)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$
For $n \in \mathbb{N}$ we set $(x_n, y_n) = (y/(n\pi), y)$. Then we have $\lim_{n\to\infty}(x_n, y_n) = (0, y)$ and
$$\partial_1 f(x_n, y_n) = 2\frac{y^2}{n\pi}\sin(n\pi) - y^2\cos(n\pi).$$
Thus $\lim_{n\to\infty} \partial_1 f(x_n, y_n)$ does not exist, and therefore $\partial_1 f$ is not continuous in $(0, y)$.

(ii) Let $y_0 \in \mathbb{R}$. From (i) we get that $J_f(0, y_0)$ is the zero 1×2 matrix. Thus the candidate for the derivative $Df(0, y_0)$ of f in $(0, y_0)$ is the constant zero function. So we must show that
$$\frac{f(x, y) - f(0, y_0)}{\sqrt{x^2 + (y - y_0)^2}}$$
tends to zero as $(x, y) \to (0, y_0)$.

We get

$$\frac{f(x,y)-f(0,y_0)}{\sqrt{x^2+(y-y_0)^2}}=\frac{x^2y\sin(\frac{y}{x})}{\sqrt{x^2+(y-y_0)^2}}.$$

Since

$$\left|\frac{x^2y\sin(\frac{y}{x})}{\sqrt{x^2+(y-y_0)^2}}\right|\le\left|\frac{x^2y}{\sqrt{x^2}}\right|\le|xy|,$$

we get that f is differentiable in $(0,y_0)$. ■