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# 12. Exercise sheet Analysis II for MCS
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(G12.1) Solution.

Since $A$ is continuous we know that also $T$ is continuous. We have

$$\frac{A(x) - A(a) - T(x-a)}{\|x-a\|} =$$

$$\frac{T(x) + c - T(a) - c - T(x) + T(a)}{\|x-a\|} = 0.$$

Thus $A'(x) = T$.

□

(G12.2) Solution.

(i) Let $g_1 : f(U) \to \mathbb{R}$ be defined by $g_1(x,y) := \sqrt{x^2+y^2}$. Then $g_1$ is continuous. Let $g_2 : f(U) \to \mathbb{R}$ be defined by for example

$$
g_2(x,y) := \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if} \quad x>0,\ y>0 \\ \frac{\pi}{2} & \text{if} \quad x=0,\ y>0 \\ \pi + \arctan\left(\frac{y}{x}\right) & \text{if} \quad x<0,\ y>0 \\ \pi + \arctan\left(\frac{y}{x}\right) & \text{if} \quad x<0,\ y\leq 0 \\ \frac{3\pi}{2} & \text{if} \quad x=0,\ y<0 \\ 2\pi + \arctan\left(\frac{y}{x}\right) & \text{if} \quad x>0,\ y<0 \end{cases}
$$

Then $g_2$ is continuous, since arctan and $(x,y) \mapsto \frac{y}{x} : \mathbb{R}^2 \setminus \{(0,y) : y \in \mathbb{R}\} \to \mathbb{R}$ are continuous and $\lim_{t\to\infty} \arctan t = \frac{\pi}{2}$, $\lim_{t\to-\infty} \arctan t = -\frac{\pi}{2}$. Furthermore,

$$
\big(g_1(f_1(r,\varphi), f_2(r,\varphi)),\ g_2(f_1(r,\varphi), f_2(r,\varphi))\big) = (r,\varphi).
$$

So $f^{-1} : f(U) \to \mathbb{R}^2$ defined by

$$
f^{-1} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}
$$

is continuous and the inverse of $f$.

(ii) The partial derivatives of $f$ are as follows:

$$\frac{\partial f_1}{\partial r}(r,\varphi) = \cos\varphi, \quad \frac{\partial f_1}{\partial \varphi}(r,\varphi) = -r\sin\varphi$$

$$\frac{\partial f_2}{\partial r}(r,\varphi) = \sin\varphi, \quad \frac{\partial f_2}{\partial \varphi}(r,\varphi) = r\cos\varphi.$$

(by Proposition 9.32)

Since $U$ is open and the partial derivatives are all continuous on $U$ it follows that $f$ is differentiable and that

$$J_f(r,\varphi) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}.$$

(Blurring the distinction between the function $f'(r,\varphi)$ and the associated matrix $J_f(r,\varphi)$ we will sometimes write e.g.

$$f'(r,\varphi) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}.)$$

(iii) The matrix in (ii) is invertible, with

$$\begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\frac{1}{r}\sin\varphi & \frac{1}{r}\cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} =: B$$

(This follows from linear algebra:
If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ad - bc \neq 0$ then
$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.)

Thus by Proposition 9.11 we know that $f^{-1}$ is differentiable (since $f'$ is continuous, $U$ and $f(U)$ are open, and $f'(r,\varphi)$ is invertible for $(r,\varphi) \in U$) with

$$(f^{-1})'(x,y) = (f'(r,\varphi))^{-1},$$

so $J_{(f^{-1})}(x,y) = B$. $\square$