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11. Exercise sheet Analysis II for MCS
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(G11.2) Solution.

First, let us show that $L(\gamma') \geq L(\gamma)$. Let $\psi : [c,d] \to [a,b]$ be the change of parameter. To any partition $P = (t_0 = a < \dots < t_n = b)$ of $[a,b]$ we associate a partition $P' = (c = t'_0 < \dots < t'_n = d)$ of $[c,d]$ by choosing for each $0 \leq i \leq n$ an arbitrary point t'_i in the set $\psi^{-1}(t_i)$. We have

$V_{P'}(\gamma') = V_P(\gamma)$. Taking the supremum over all partitions of $[c,d]$, we obtain $L(\gamma') \geq V_P(\gamma)$. Now taking the supremum over all partitions P of $[a,b]$, we obtain $L(\gamma') \geq L(\gamma)$.

For the converse inequality, let us consider a partition $P = (c = t_0 < \dots < t_n = d)$ of $[c,d]$. Its image $\psi(\{t_0, \dots, t_n\})$ gives rise to a partition P' of $[a,b]$ satisfying $V_P(\gamma') = V_{P'}(\gamma)$, since ψ is monotone. (Here $P' = (a = \psi(t_0) < \ldots < \psi(t_n) = b)$ or $P' = (a = \psi(t_n) < \ldots < \psi(t_0) = b)$, with possibly fewer than $n+1$ elements, since we might have $\psi(t_i) = \psi(t_j)$ for $i \neq j$.) Taking the supremum over all partitions of $[a,b]$ we obtain $V_P(\gamma') \leq L(\gamma)$. Hence by taking the supremum over all partitions $[c,d]$, we get that $L(\gamma') \leq L(\gamma)$. Thus we have $L(\gamma') = L(\gamma)$.

$\square$

(G 11.3) Solution.

We remark that if P, P' are partitions of $[a,b]$ with P' a refinement of P, then the triangle inequality yields by induction that $V_P(\gamma) \leq V_{P'}(\gamma)$. We assume that $a < c < b$, for else the proof is trivial. Let $P_1 = (a = t_0 < \ldots < t_n = c)$ be a partition of $[a,c]$ and let $P_2 = (c = t'_0 < \ldots < t'_m = b)$

be a partition of $[c,b]$. Write $t_{i+n} := t'_i$ for $0 \le i \le m$. Then $P_2 = (c = t_n < \ldots < t_{n+m} = b)$, and we have that $P = (a = t_0 < \ldots < t_n < \ldots < t_{n+m} = b)$ is a partition of $[a,b]$. So

$$V_{P_1}(\gamma|_{[a,c]}) + V_{P_2}(\gamma|_{[c,b]}) =$$

$$\sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) + \sum_{i=n}^{n+m-1} d(\gamma(t_i), \gamma(t_{i+1}))$$

$$= \sum_{i=0}^{n+m-1} d(\gamma(t_i), \gamma(t_{i+1}))$$

$$= V_P(\gamma).$$

Thus if $\sup\{V_P(\gamma|_{[a,c]}) : P \text{ is a partition of } [a,c]\}$ or $\sup\{V_P(\gamma|_{[c,b]}) : P \text{ is a partition of } [c,b]\}$ does not exist, then $\sup\{V_P(\gamma) : P \text{ is a partition of } [a,b]\}$ does not exist. That is, if $L(\gamma|_{[a,c]}) = \infty$ or $L(\gamma|_{[c,b]}) = \infty$, then $L(\gamma) = \infty$.

If $L(\gamma) < \infty$, then we get $V_{P_1}(\gamma|_{[a,c]}) + V_{P_2}(\gamma|_{[c,b]}) \le L(\gamma)$.

So $\quad L(\gamma|_{[a,c]}) + L(\gamma|_{[c,b]}) \le L(\gamma)$.

Let now $P = (a = t_0 < \ldots < t_k = b)$ be a partition of $[a,b]$. We assume that $t_m = c$

for some $m \leq k$, for else we consider a refinement of P where this holds. So

$$P = (a = t_0 < \ldots < t_m = c < \ldots < t_{m+n} = b)$$

for some $n > 0$. Let now

$P_1 = (a = t_0' < \ldots < t_m' = c)$ be a partition of $[a,c]$ with $t_i' := t_i$ for $i \leq m$, and let

$P_2 = (c = t_0'' < \ldots < t_n'' = b)$ be a partition of $[c,b]$ with $t_i'' := t_{m+i}$ for $i \leq n$. Then

$$V_P(\gamma) = \sum_{i=0}^{m+n-1} d(\gamma(t_i), \gamma(t_{i+1}))$$

$$= \sum_{i=0}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})) + \sum_{i=0}^{n-1} d(\gamma(t_{m+i}), \gamma(t_{m+i+1}))$$

$$= \sum_{i=0}^{m-1} d(\gamma(t_i'), \gamma(t_{i+1}')) + \sum_{i=0}^{n-1} d(\gamma(t_i''), \gamma(t_{i+1}''))$$

$$= V_{P_1}(\gamma|_{[a,c]}) + V_{P_2}(\gamma|_{[c,b]}).$$

So if $\sup\{V_P(\gamma) : P \text{ is a partition of } [a,b]\}$ does not exist, then either

$\sup\{V_P(\gamma|_{[a,c]}) : P \text{ is a partition of } [a,c]\}$

or

$\sup\{V_P(\gamma|_{[c,b]}) : P \text{ is a partition of } [c,b]\}$

does not exist. That is, if $L(\gamma) = \infty$, then $L(\gamma|_{[a,c]}) = \infty$ or $L(\gamma|_{[c,b]}) = \infty$.

If $\underline{L(\gamma|_{[a,c]}) < \infty \text{ and }} L(\gamma|_{[c,b]}) < \infty$, then $V_P(\gamma) \le L(\gamma|_{[a,c]}) + L(\gamma|_{[c,b]})$. And so

$$L(\gamma) \le L(\gamma|_{[a,c]}) + L(\gamma|_{[c,b]}).$$

Thus $L(\gamma) = L(\gamma|_{[a,c]}) + L(\gamma|_{[c,b]})$.

(Where we interpret this to mean ordinary equality between real numbers if $L(\gamma)$, $L(\gamma|_{[a,c]})$ and $L(\gamma|_{[c,b]})$ are real numbers, and in addition

$$L(\gamma) = \infty \Leftrightarrow \left(L(\gamma|_{[a,c]}) = \infty \vee L(\gamma|_{[c,b]}) = \infty\right).$$

$L(\gamma) = \infty$ means that

$$\sup \{ V_P(\gamma) : P \text{ is a partition of } [a,b] \}$$

does not exist. $L(\gamma) < \infty$ means that this supremum exists and is a real number.)

$\square$