Fachbereich Mathematik
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10. Exercise sheet Analysis II for MCS
Summer Term 2006

(G10.1)

For each $n \in \mathbb{N}$ define $f_n : [0, \infty[\to \mathbb{R}$ by $f_n(x) := x^n/(1 + x^n)$.

(i) Show that f_n is bounded, for each $n \in \mathbb{N}$.

(ii) Show that the sequence $(f_n)_n$ converges uniformly on the interval $[0, c]$ for any number $0 < c < 1$.

(iii) Show that the sequence $(f_n)_n$ converges uniformly on the interval $[b, \infty[$ for $b > 1$, but not on the interval $[1, \infty[$.

Solution. Handwritten.

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(G10.2)

Let $(V, \|\cdot\|)$ be a normed space. For a non-zero element $x \in V$ we say that $x/\|x\|$ is the *normalized* element corresponding to x. We then denote $x/\|x\|$ by $u(x)$.

Let $x, y \in V$ be non-zero. Prove that

$$\|u(x) - u(y)\| \le 2\frac{\|x - y\|}{\|x\|}.$$

Solution. Handwritten.

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(G10.3) (Supplementary exercise)

Prove Dini's Theorem:

Let X be a compact metric space. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions with $f_n : X \to \mathbb{R}$ for each $n \in \mathbb{N}$. Suppose that for each $x \in X$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ is increasing and bounded. Let $f : X \to \mathbb{R}$ be the pointwise limit of $(f_n)_{n\in\mathbb{N}}$, i.e.

$$f(x) = \sup_{n\in\mathbb{N}} f_n(x)$$

for all $x \in X$. Suppose further that f is continuous. Then the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

Solution.

Let $x \in X$ and $\varepsilon > 0$. Then there is an index $N_x \in \mathbb{N}$ such that

$$f(x) - \varepsilon < f_{N_x}(x).$$

Since f and f_{N_x} are continuous, there is a neighborhood U_x of x such that

$$f(y) - \varepsilon < f_{N_x}(y)$$

for all $y \in U_x$. It is obvious that $X = \bigcup_{x\in X} U_x$. By the compactness of X there is $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in X$ such that $X = \bigcup_{i=1}^{k} U_{x_i}$. Let now $N = \max\{N_{x_i} : 1 \le i \le k\}$. This implies that for all $n \ge N$ and all $x \in X$

$$f(x) - \varepsilon < f_N(x) \le f_n(x) \le f(x),$$

and this completes the proof.

Notice that if we in the theorem take X to be a compact topological space instead of a compact metric space, then the above proof is also a proof of this stronger statement.

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