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10 Exercise sheet Analysis II for MCS
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(G 10.1) Solution.

(i) For $x \geq 0$ we have $x^n \leq 1 + x^n$ and so $0 \leq f_n(x) \leq 1$.

(ii) For $x \in [0, c]$ we have

$$\left|\frac{x^n}{1+x^n}\right| \leq |x^n| \leq c^n.$$

Since $c^n \to 0$ as $n \to \infty$ we get that (f_n) converges uniformly on $[0, c]$.

(iii) If $x \geq b > 1$, then

$$f_n(x) - 1 = \frac{x^n}{1+x^n} - \frac{1+x^n}{1+x^n} = \frac{-1}{1+x^n}$$

gives

$$|f_n(x) - 1| = \frac{1}{|1+x^n|} \leq \frac{1}{|x|^n - 1} \leq \frac{1}{b^n - 1}.$$

Since $b^n \to \infty$ as $n \to \infty$ we get that (f_n) converges uniformly on $[b, \infty[$ to 1.

For $x > 1$ we have $\lim_{n\to\infty} f_n(x) = 1$.

Furthermore we have $f_n(1) = \frac{1}{2}$ for all n.

Hence $f : [1, \infty[\to \mathbb{R}$, $f(x) := \lim_{n\to\infty} f_n(x)$ is not continuous and so the convergence on $[1, \infty[$ can not be uniform, since each f_n is continuous.

$\square$

(G10.2) Solution.

Let $z = \|x\| \cdot \|y\| (u(x) - u(y)) = \|y\| x - \|x\| y$.

Then
$$z = \|y\| (x-y) - (\|x\| - \|y\|) y$$
$$= \|y\| (x-y) + (\|y\| - \|x\|) y.$$

Thus
$$\|z\| \le \|y\| \|x-y\| + \big| \|y\| - \|x\| \big| \|y\|$$
$$\le \|y\| \|x-y\| + \|y\| \|x-y\|,$$

since $\big| \|y\| - \|x\| \big| \le \|x-y\|$. (This follows since $\|x-y\| + \|y\| \ge \|x\|$ gives $\|x-y\| \ge \|x\| - \|y\|$ and $\|x-y\| = \|y-x\|$ and $\|y-x\| + \|x\| \ge \|y\|$ gives $\|x-y\| \ge \|y\| - \|x\|$.)

Now $\|z\| \leq 2\|y\|\,\|x-y\|$ gives

$$\|u(x) - u(y)\| \leq \frac{2\,\|x-y\|}{\|x\|}$$

from the definition of z.

$\square$