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9. Exercise sheet Analysis II for MCS
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(G9.1) Solution.

Let $c, C \in \,]0,\infty[$ be such that

$$(\forall x \in V)\,(c\,\|x\|_1 \le \|x\|_2 \le C\,\|x\|_1).$$

Then $(\forall x \in V)\,\left(\frac{1}{C}\,\|x\|_2 \le \|x\|_1 \le \frac{1}{c}\,\|x\|_2\right)$,
so the relation is symmetric.

We have

$$(\forall x \in V)\,(1\cdot\|x\|_1 \le \|x\|_1 \le 1\cdot\|x\|_1),$$

so the relation is reflexive.

Suppose $c, C, c_1, C_1 \in \,]0,\infty[$ are such that

$$(\forall x \in V)\,(c\,\|x\|_1 \le \|x\|_2 \le C\,\|x\|_1),$$

$$(\forall x \in V)\,(c_1\|x\|_2 \le \|x\|_3 \le C_1\|x\|_2),$$

i.e. $\|\cdot\|_1$ and $\|\cdot\|_2$, and $\|\cdot\|_2$ and $\|\cdot\|_3$, are equivalent. Let $x \in V$.

Since $\|x\|_2 \leq C\|x\|_1$ and $\|x\|_3 \leq C_1\|x\|_2$ we get $\|x\|_3 \leq CC_1\|x\|_1$. Also, since $c_1\|x\|_2 \leq \|x\|_3$ and $c\|x\|_1 \leq \|x\|_2$ we get $cc_1\|x\|_1 \leq \|x\|_3$. So

$$(\forall x \in V)\left(cc_1\|x\|_1 \leq \|x\|_3 \leq CC_1\|x\|_1\right),$$

and the relation is thus transitive.

Hence equivalence of norms is an equivalence relation.

$\square$

(G 9.2) Solution.

(i) Suppose $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_1)$.

Then

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m,n \geq N)\left(\|x_m - x_n\|_1 < \varepsilon\right).$$

Since $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent we can let $C \in]0,\infty[$ be such that

$$(\forall x \in V)\left(\|x\|_2 \leq C\|x\|_1\right).$$

Let $\varepsilon > 0$, and let $N \in \mathbb{N}$ be such that

$$(\forall m,n \geq N)\left(\|x_m - x_n\|_1 < \frac{\varepsilon}{C}\right).$$

Then for $m, n \geq N$ we have

$$\|x_m - x_n\|_2 \leq C\|x_m - x_n\|_1 < \varepsilon .$$

Thus $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_2)$.

The same argument applies to show that if $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_2)$ then $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $(V, \|\cdot\|_1)$.

(ii) Let $\lim_{n\to\infty} x_n = x$ in $(V, \|\cdot\|_1)$, i.e.

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)(\|x - x_n\|_1 < \varepsilon).$$

Let $C \in]0, \infty[$ be s.t.

$$(\forall x \in V)(\|x\|_2 \leq C\|x\|_1).$$

Then for $\varepsilon > 0$ we let $N \in \mathbb{N}$ be such that

$$(\forall n \geq N)\left(\|x - x_n\|_1 < \frac{\varepsilon}{C}\right).$$

So for $n \geq N$ we have

$$\|x - x_n\|_2 \leq C\|x - x_n\|_1 < \varepsilon .$$

Thus $\lim_{n\to\infty} x_n = x$ in $(V, \|\cdot\|_2)$.

(iii) Let $A \subseteq V$ be open in $(V, \|\cdot\|_1)$, i.e. $V \setminus A$ is closed in $(V, \|\cdot\|_1)$. So $V \setminus A$ contains all of its accumulation points with respect to the metric induced by $\|\cdot\|_1$.

Let $x \in V$ be an accumulation point of $V \setminus A$ with respect to the metric induced by $\|\cdot\|_2$. We must show $x \in V \setminus A$, for then $V \setminus A$ is closed in $(V, \|\cdot\|_2)$, so A is open in $(V, \|\cdot\|_2)$. We have

$$(\forall \varepsilon > 0)((\{ y : y \in V \wedge \|x-y\|_2 < \varepsilon\} \setminus \{x\}) \cap (V \setminus A) \neq \emptyset).$$

Let $C \in]0, \infty[$ be s.t.

$$(\forall x \in V)(\|x\|_1 \leq C \|x\|_2),$$

and let $\varepsilon > 0$. Let further $y \in V \setminus A$ s.t. $y \neq x$, $\|x-y\|_2 < \frac{\varepsilon}{C}$. Then

$$\|x-y\|_1 \leq C \|x-y\|_2 < \varepsilon, \quad \text{so}$$

$$(\forall \varepsilon > 0)((\{ y : y \in V \wedge \|x-y\|_1 < \varepsilon\} \setminus \{x\}) \cap (V \setminus A) \neq \emptyset).$$

Hence x is an accumulation point of $V \setminus A$ with respect to the metric induced by $\|\cdot\|_1$,

and so $x \in V \setminus A$. Thus $V \setminus A$ is closed in $(V, \|\cdot\|_2)$, so A is open in $(V, \|\cdot\|_2)$.
The same argument applies to show that if A is open in $(V, \|\cdot\|_2)$ then A is open in $(V, \|\cdot\|_1)$.

Assume now that $A \subseteq V$ is bounded in $(V, \|\cdot\|_1)$. That is, there exists $M > 0$ s.t. $(\forall x, y \in A)(\|x - y\|_1 < M)$.
Let $C \in]0, \infty[$ be s.t.

$$(\forall x \in V)(\|x\|_2 \leq C\|x\|_1).$$

Then for $x, y \in A$

$$\|x - y\|_2 \leq C\|x - y\|_1 < CM,$$

so

$$(\forall x, y \in A)(\|x - y\|_2 < M')$$

with $M' = CM$. Thus A is bounded in $(V, \|\cdot\|_2)$.

(G9.3) Solution.

Let $f:[a,b] \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{else}. \end{cases}$$

Then f is Riemann integrable, and $\int_a^b f = 0$. Furthermore, for $1 \le p < \infty$

$|f|^p = f$, so

$$\left(\int_a^b |f|^p \right)^{1/p} = 0.$$

Thus $\|f\|_p = 0$, but $f \neq 0$, so $\|\cdot\|_p$ is not a norm on $I([a,b])$.

$\square$