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8. Exercise sheet Analysis II for MCS
Summer Term 2006.

(G8.1) Solution.

(i) We must prove that $\|\cdot\|_\infty$ and $\|\cdot\|_p$ $(1 \le p < \infty)$ fulfill N1, N2, N3 from definition 6.1 in the handouts.

We check N1:

Firstly, we have $\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\} \ge 0$ since $|x_1|, \dots, |x_n| \ge 0$. We also have that $\|x\|_\infty = 0$ gives $|x_1|, \dots, |x_n| = 0$, and so $x = 0$. If $x = 0$ then $|x_1|, \dots, |x_n| = 0$, and so $\|x\|_\infty = 0$.

We have $\|x\|_p \ge 0$ since $\sum_{i=1}^{n} |x_i|^p \ge 0$, which follows since $|x_1|, \dots, |x_n| \ge 0$.

If $\|x\|_p = 0$ then $\sum_{i=1}^{n} |x_i|^p = 0$, and so

$|x_1|, \dots, |x_n| = 0$. So $x = 0$. If on the other hand $x=0$, then $|x_1|, \dots, |x_n| = 0$, and so $\sum_{i=1}^{n} |x_i|^p = 0$. So $\|x\|_p = 0$.

Thus $\|\cdot\|_\infty$ and $\|\cdot\|_p$ fulfill N1.

Now we check N2. Let $\lambda \in \mathbb{K}$ and let $x \in \mathbb{K}^n$. We have

$$\|\lambda \cdot x\|_\infty = \max\{|\lambda \cdot x_1|, \dots, |\lambda \cdot x_n|\} = \max\{|\lambda| \cdot |x_1|, \dots, |\lambda| \cdot |x_n|\}$$

$$= |\lambda| \max\{|x_1|, \dots, |x_n|\} = |\lambda| \cdot \|x\|_\infty .$$

And also

$$\|\lambda \cdot x\|_p = \left(\sum_{i=1}^{n} |\lambda \cdot x_i|^p\right)^{1/p} = \left(\sum_{i=1}^{n} |\lambda|^p \cdot |x_i|^p\right)^{1/p}$$

$$= \left(|\lambda|^p \sum_{i=1}^{n} |x_i|^p\right)^{1/p} = |\lambda| \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} = |\lambda| \cdot \|x\|_p .$$

So $\|\cdot\|_\infty$ and $\|\cdot\|_p$ satisfy N2.

Finally, we prove that $\|\cdot\|_\infty$ and $\|\cdot\|_p$ fulfill N3, i.e. the triangle inequality. Let $x, y \in \mathbb{K}^n$.

Then

$$\|x+y\|_\infty = \max\{|x_1+y_1|, \dots, |x_n+y_n|\} \leq \max\{|x_1|+|y_1|, \dots, |x_n|+|y_n|\}$$

$$\leq \max\{|x_1|, \dots, |x_n|\} + \max\{|y_1|, \dots, |y_n|\}$$

$$= \|x\|_\infty + \|y\|_\infty.$$

Also

$$\|x+y\|_p = \left(\sum_{i=1}^n |x_i+y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^n (|x_i|+|y_i|)^p\right)^{1/p},$$

and by the Minkowski inequality from (G 5.3) we get

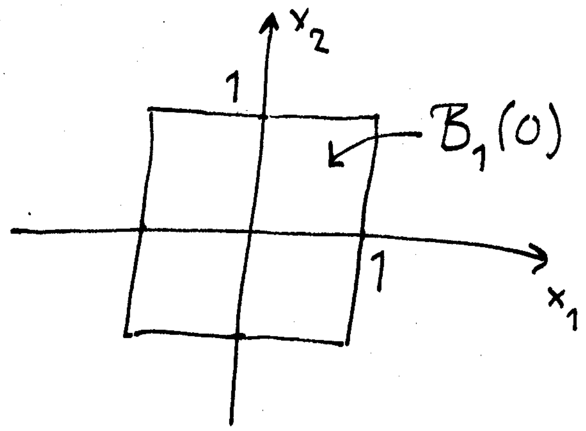
$$\left(\sum_{i=1}^n (|x_i|+|y_i|)^p\right)^{1/p} \leq \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^n |y_i|^p\right)^{1/p}$$

$$= \|x\|_p + \|y\|_p.$$

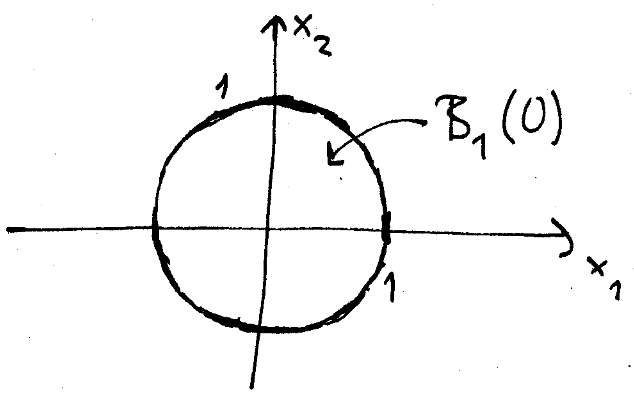
Thus $\|x+y\|_p \leq \|x\|_p + \|y\|_p$.

We have proved that both $\|\cdot\|_\infty$ and $\|\cdot\|_p$ satisfy N1, N2 and N3.

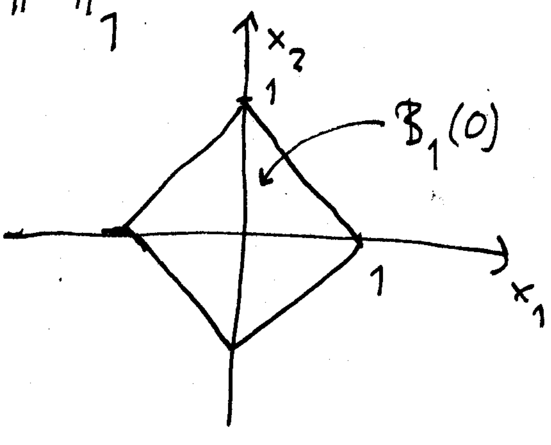
(ii) The maximum norm $\|\cdot\|_\infty$:



The Euclidean norm $\|\cdot\|_2$:



1-norm $\|\cdot\|_1$:



$\square$

(G8.2) Solution.

(i) We have $\|(1,-1,0)\| = 1+(-1)+0 = 0$,

but $(1,-1,0)$ is not the zero vector,

so N1 is not satisfied.

(ii) We have $\|(x_1, x_2, x_3)\| = |x_1| + |x_2| + 2|x_3| \geq 0$,

$\|(0,0,0)\| = 0$, and if

$\|(x_1, x_2, x_3)\| = 0$ then $|x_1|, |x_2|, |x_3| = 0$

and so $x_1, x_2, x_3 = 0$. So N1 is satisfied.

Let $\lambda \in \mathbb{R}$. Then

$$\|(\lambda x_1, \lambda x_2, \lambda x_3)\| = |\lambda x_1| + |\lambda x_2| + 2|\lambda x_3|$$

$$= |\lambda|\,|x_1| + |\lambda|\,|x_2| + |\lambda|\,2\,|x_3|$$

$$= |\lambda|\,\|(x_1, x_2, x_3)\|,$$

so N2 is satisfied.

Let $(x_1, x_2, x_3) \in \mathbb{R}^3$ and $(y_1, y_2, y_3) \in \mathbb{R}^3$. Then

$$\|(x_1+y_1, x_2+y_2, x_3+y_3)\| = |x_1+y_1| + |x_2+y_2| + 2|x_3+y_3|$$

$$\leq (|x_1| + |y_1|) + (|x_2| + |y_2|) + 2(|x_3| + |y_3|)$$

$$= (|x_1| + |x_2| + 2|x_3|) + (|y_1| + |y_2| + 2|y_3|)$$

$$= \|(x_1, x_2, x_3)\| + \|(y_1, y_2, y_3)\|.$$

So also N3 is satisfied, and $\|\cdot\|$ is thus a norm on $\mathbb{R}^3$.

(iii) We write $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Since $x_i^2 = |x_i|^2$ for $x_i \in \mathbb{R}$, we notice that we have

$$\|x\| = 2\|x\|_2,$$

where $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{R}^3$.

Thus $\|x\| \geq 0$ and $(\|x\| = 0 \Leftrightarrow \|x\|_2 = 0 \Leftrightarrow x = 0)$, so N1 is satisfied. Let $\lambda \in \mathbb{R}$.

Then

$$\|\lambda x\| = 2\|\lambda x\|_2 = 2|\lambda|\,\|x\|_2 = |\lambda|\,\|x\|,$$ so

N2 holds.

Let $x, y \in \mathbb{R}^3$. Then

$$\|x+y\| = 2\|x+y\|_2 \leq 2(\|x\|_2 + \|y\|_2)$$

$$= 2\|x\|_2 + 2\|y\|_2 = \|x\| + \|y\|,$$

so N3 holds. Thus $\|\cdot\|$ is a norm on $\mathbb{R}^3$.

$\square$

(G8.3) Solution.

(i) Since $\|v\|_1 \geq 0$ and $\|v\|_2 \geq 0$ for any $v \in V$, we get

$$\|v\| = \|v\|_1 + \|v\|_2 \geq 0.$$ Thus also:

$$\|v\| = 0 \iff (\|v\|_1 = 0 \text{ and } \|v\|_2 = 0) \iff v = 0.$$

So N1 holds. Let λ be a scalar.

Then $\|\lambda v\| = \|\lambda v\|_1 + \|\lambda v\|_2 = |\lambda|\,\|v\|_1 + |\lambda|\,\|v\|_2$

$= |\lambda|\,\|v\|$, so N2 holds.

Let $x, y \in V$. Then

$$\|x+y\| = \|x+y\|_1 + \|x+y\|_2 \leq (\|x\|_1 + \|y\|_1) + (\|x\|_2 + \|y\|_2)$$

$$= (\|x\|_1 + \|x\|_2) + (\|y\|_1 + \|y\|_2) = \|x\| + \|y\|,$$

thus N3 holds, and so $\|\cdot\|$ is a norm on V.

(ii) If $a = 0$ then $a\|\cdot\|$ is not a norm unless V is the vector space with only the zero vector, i.e. $V = \{0\}$. Because then we would have a nonzero vector v with ~~[illegible]~~ $a\|v\| = 0$.

If $a < 0$, then $a\|\cdot\|$ again is not a norm unless $V = \{0\}$. We would then have a nonzero vector $v \in V$ with $\|v\| > 0$ and thus $a \cdot \|v\| < 0$.

If $V = \{0\}$ then $\|v\| = 0$ for all $v \in V$, and so $a\|\cdot\|$ is a norm for any $a \in \mathbb{R}$, since $a\|0\| = 0$. (It is trivial to check N1, N2, N3.)

If $a > 0$ then $a\|\cdot\|$ is a norm on V. The argument is completely parallel to the argument in the solution to (G8.2)(iii).

$\square$