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7. Exercise sheet Analysis II for MCS
SS 2006. Solutions.

(G7. 1) (i) We have $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$, so

$$\int_0^{1/2} \arcsin x \, dx = \Big[x \arcsin x\Big]_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\, dx.$$

By substituting $t = 1-x^2$, $dt = -2x\,dx$, we get

$$\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\, dx = -\frac{1}{2}\int_1^{3/4} \frac{dt}{\sqrt{t}}, \quad \text{so}$$

$$\int_0^{1/2} \arcsin x \, dx = \Big[x \arcsin x\Big]_0^{1/2} + \frac{1}{2}\int_1^{3/4} \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2}\arcsin\frac{1}{2} + \Big[\sqrt{t}\Big]_1^{3/4} = \frac{1}{2}\arcsin\frac{1}{2} + \sqrt{\frac{3}{4}} - 1$$

$$\left(= \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1 \right)$$

(G.7.1)(ii) We use substitution.

$$\int_0^1 \frac{6x^2+4}{x^3+2x+1}\,dx = 2\cdot\int_0^1 \frac{(x^3+2x+1)'}{x^3+2x+1}\,dx$$

$$= 2\cdot\int_1^4 \frac{1}{t}\,dt = 2\cdot\Big[\log(t)\Big]_1^4 = 2\cdot\log(4).$$

iii) We note that $1-x^2=(1-x)(1+x)$, and try to find $\alpha, \beta \in \mathbb{R}$ such that

$$\frac{1}{1-x^2} = \frac{\alpha}{1-x} + \frac{\beta}{1+x},$$

that is,

$$\frac{1}{1-x^2} = \frac{(\alpha+\beta)+(\alpha-\beta)x}{1-x^2}.$$

So $\alpha = \beta = \frac{1}{2}$.

We get

$$\int_a^b \frac{dx}{1-x^2} = \frac{1}{2}\left(\int_a^b \frac{dx}{1-x} + \int_a^b \frac{dx}{1+x}\right) =$$

$$= \frac{1}{2}\left(\int_a^b \frac{dx}{x+1} - \int_a^b \frac{dx}{x-1} \right)$$

$$= \frac{1}{2}\Big[\big(\ln|x+1| - \ln|x-1|\big)\Big]_a^b$$

$$= \frac{1}{2}\left[\ln\left|\frac{x+1}{x-1}\right|\right]_a^b$$

$$= \frac{1}{2}\ln\left|\frac{b+1}{b-1}\right| - \frac{1}{2}\ln\left|\frac{a+1}{a-1}\right| .$$

(G 7.2) Integration by parts gives for $m \geq 2$ that
$$I_m = -\int_a^b \sin^{m-1} x \cdot (\cos x)' \, dx$$
$$= -\Big[\cos x \cdot \sin^{m-1} x\Big]_a^b + (m-1)\int_a^b \cos^2 x \cdot \sin^{m-2} x \, dx$$
$$= -\Big[\cos x \cdot \sin^{m-1} x\Big]_a^b + (m-1)\int_a^b (1-\sin^2 x)\sin^{m-2} x \, dx$$
$$= -\Big[\cos x \cdot \sin^{m-1} x\Big]_a^b + (m-1)\, I_{m-2} - (m-1)\, I_m .$$

So we get
$$I_m = -\frac{1}{m}\Big[\cos x \cdot \sin^{m-1} x\Big]_a^b + \frac{m-1}{m} I_{m-2} .$$

Since
$$I_0 = \int_a^b \sin^0 x \, dx = b - a$$
and
$$I_1 = \int_a^b \sin x \, dx = \cos a - \cos b ,$$
we can recursively calculate I_m with this formula.