Fachbereich Mathematik
Dr. L. Leuştean
E. Briseid, S. Herrmann

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6. Exercise sheet Analysis II for MCS
Summer Term 2006

(G6.1)

Prove that the characteristic function of the rational numbers in the unit interval, i.e. the function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q},\\ 0 & \text{otherwise,}\end{cases}$$

is not Riemann integrable.

Solution.

Since f is bounded, the upper and the lower integrals of f exist. We shall prove that they are not equal. Let S_f be the set of all step functions $s\in S[0,1]$ with $s\le f$, and let S^f be the set of all step functions $t\in S[0,1]$ with $f\le t$. Then

$$\underline{\int} f=\sup\{\int s\mid s\in S_f\},\quad \overline{\int} f=\inf\{\int t\mid t\in S^f\}.$$

First, we shall prove that $\int s\le 0$ for any $s\in S_f$, and that $\int t\ge 1$ for any $t\in S^f$. Let $s\in S_f, t\in S^f$. Then there is a partition $P=(0=x_0<\ldots<x_n=1)$ such that s takes constant values s_i and t takes constant values t_i on each interval $]x_{i-1},x_i[$, for $1\le i\le n$. Each of these intervals contain both a rational number q_i and an irrational number r_i. Thus, we get

$$s_i=s(r_i)\le f(r_i)=0,\quad t_i=t(q_i)\ge f(q_i)=1,$$

so

$$\int s=\sum_{i=1}^{n}s_i(x_i-x_{i-1})\le 0,\quad \int t=\sum_{i=1}^{n}t_i(x_i-x_{i-1})\ge 1.$$

Hence,

$$\underline{\int} f\le 0,\quad \overline{\int} f\ge 1.$$

Since the function $s^*:[0,1]\to\mathbb{R}$, $s^*(x)=0$ is in S_f and $\int s=0$, it follows that $\underline{\int} f=0$.

Similarly, the function $t^*:[0,1]\to\mathbb{R}$, $t^*(x)=1$ is in S^f and $\int t=1$, hence $\overline{\int} f=1$.

Thus, $\underline{\int} f\ne\overline{\int} f$, so f is not Riemann integrable. ■

(G6.2)

(i) Let $a,b\in\mathbb{R}$ with $a<b$, and let $f:[a,b]\to\mathbb{R}$ be a continuous function such that $\int_a^b f(x)\,dx=0$. Prove that there is $c\in[a,b]$ such that $f(c)=0$.

(ii) Let $a<b\in\mathbb{R}$ and let $f:[a,b]\to\mathbb{R}$ be an isotone function. Prove that

$$f(a)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le f(b).$$

Solution.

(i) Assume that $f(c)\ne 0$ for all $c\in[a,b]$. Since f is continuous, we get that f has constant sign on $[a,b]$. Assume that $f>0$. Then $m:=\min\{f(x)\mid x\in[a,b]\}>0$. Since $f(x)\ge m$ for all $x\in[a,b]$, it follows that

$$\int_a^b f(x)\,dx\ge\int_a^b m\,dx=m(b-a)>0,$$

which contradicts the hypothesis.
Similarly, if $f<0$, then $M:=\max\{f(x)\mid x\in[a,b]\}<0$, so

$$\int_a^b f(x)\,dx\le\int_a^b M\,dx=M(b-a)<0,$$

again a contradiction with the hypothesis.

(ii) Since f is isotone, we have that $f(a)\le f(x)\le f(b)$ for all $x\in[a,b]$. It follows that

$$(b-a)f(a)=\int_a^b f(a)\,dx\le\int_a^b f(x)\,dx\le\int_a^b f(b)\,dx=(b-a)f(b),$$

that is,

$$f(a)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le f(b).$$

■

(G6.3) (Supplementary exercise)

Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{n\to\infty} \int_0^1 x^n f(x)\,dx = 0.$$

Solution.

Since f is continuous, we get that f is also bounded, so there is $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in [0,1]$. It follows that

$$\begin{aligned}\left|\int_0^1 x^n f(x)\,dx\right| &\leq \int_0^1 |x^n f(x)|\,dx = \int_0^1 x^n |f(x)|\,dx \leq \int_0^1 x^n \cdot M\,dx = M \cdot \int_0^1 x^n\,dx \\ &= M \cdot \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{M}{n+1}.\end{aligned}$$

Since $\lim_{n\to\infty} \frac{M}{n+1} = 0$, we get that $\lim_{n\to\infty} \left|\int_0^1 x^n f(x)\,dx\right| = 0$, so $\lim_{n\to\infty} \int_0^1 x^n f(x)\,dx = 0$. ■

Orientation Colloquium

The Department of Mathematics' Research Groups present themselves.

Monday, 29.05.2006 – 16:15-17:15 – S207/109

Prof. Dr. Burkhard Kümmerer

FG Algebra, Geometrie und Funktionalanalysis

"Im Dreiländereck Funktionalanalysis – Stochastik – Mathematische Physik"

After the talk there will be a relaxed get-together (coffee, tea and biscuits) in S215/219, where interested people can discuss the talk and become more acquainted with the lecturer.