Fachbereich Mathematik
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5. Exercise sheet Analysis II for MCS
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(G5.1)

Let $f, g : [a, b] \to \mathbb{R}$ be bounded real functions. We define

$$\|f\| = \sup\{|f(x)| : x \in [a, b]\}.$$

Prove the following statements:

(i) $\|f\| \geq 0$ and ($\|f\| = 0$ if and only if $f = 0$).

(ii) $\|f + g\| \leq \|f\| + \|g\|$.

(iii) $\|fg\| \leq \|f\| \cdot \|g\|$. In general $\|fg\| \neq \|f\| \cdot \|g\|$.

Solution.

(i) Since $|f(x)| \geq 0$ for all $x \in [a, b]$, we have $\|f\| = \sup\{|f(x)| : x \in [a, b]\} \geq 0$. If $f = 0$ then obviously $\|f\| = 0$. If $\|f\| = 0$, then $|f(x)| = 0$ for all $x \in [a, b]$ and therefore $f = 0$.

(ii) For each $x \in [a, b]$ we have $|f(x) + g(x)| \leq |f(x)| + |g(x)|$, hence

$$|f(x) + g(x)| \leq \sup\{|f(x)| : x \in [a, b]\} + \sup\{|g(x)| : x \in [a, b]\} = \|f\| + \|g\|.$$

Thus,
$$\|f + g\| = \sup\{|f(x) + g(x)| : x \in [a, b]\} \leq \|f\| + \|g\|.$$

(iii) For each $x \in [a, b]$ we have

$$|f(x) \cdot g(x)| = |f(x)| \cdot |g(x)| \leq \sup\{|f(x)| : x \in [a, b]\} \cdot \sup\{|g(x)| : x \in [a, b]\} = \|f\| \cdot \|g\|.$$

Thus,
$$\|fg\| = \sup\{|f(x) \cdot g(x)| : x \in [a, b]\} \leq \|f\| \cdot \|g\|.$$

Now we consider the step functions $f, g : [a, b] \to \mathbb{R}$ with

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}], \\ 1 & \text{if } x \in]\frac{1}{2}, 1], \end{cases}$$

and

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}], \\ 0 & \text{if } x \in]\frac{1}{2}, 1]. \end{cases}$$

Then $fg = 0$, so $\|fg\| = 0$. On the other hand, $\|f\| = 1 = \|g\|$, i.e. $\|f\| \cdot \|g\| = 1$. ■

(G5.2)

Use the arithmetical-geometrical inequality in (G3.2) (ii) to prove the **Hölder inequality**:

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q},$$

for all $n \in \mathbb{N}$ and for all $a_i, b_i \geq 0$ for $1 \leq i \leq n$, and for all $p, q \in]1, \infty[$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Solution.

We can assume that $\sum_{i=1}^n a_i \neq 0$ and $\sum_{i=1}^n b_i \neq 0$, for else the proof is trivial. For $1 \leq i \leq n$, we let

$$\alpha_i := \frac{a_i^p}{\sum_{j=1}^n a_j^p},$$

and

$$\beta_i := \frac{b_i^q}{\sum_{j=1}^n b_j^q}.$$

Then $\sum_{i=1}^n \alpha_i = 1$ and $\sum_{i=1}^n \beta_i = 1$. We have

$$\frac{a_i b_i}{\left(\sum_{j=1}^n a_j^p\right)^{1/p} \left(\sum_{j=1}^n b_j^q\right)^{1/q}} = \alpha_i^{1/p} \beta_i^{1/q}.$$

If $\alpha_i = 0$ or $\beta_i = 0$, then

$$\alpha_i^{1/p} \beta_i^{1/q} \leq \frac{\alpha_i}{p} + \frac{\beta_i}{q}.$$

If $\alpha_i > 0$ and $\beta_i > 0$, we apply the arithmetical-geometrical inequality from (G3.2) (ii) and get

$$\alpha_i^{1/p} \beta_i^{1/q} \leq \frac{\alpha_i}{p} + \frac{\beta_i}{q}.$$

Hence

$$\frac{\sum_{i=1}^n a_i b_i}{\left(\sum_{j=1}^n a_j^p\right)^{1/p}\left(\sum_{j=1}^n b_j^q\right)^{1/q}} = \sum_{i=1}^n \left(\frac{a_i b_i}{\left(\sum_{j=1}^n a_j^p\right)^{1/p}\left(\sum_{j=1}^n b_j^q\right)^{1/q}}\right) \leq \sum_{i=1}^n \frac{\alpha_i}{p} + \frac{\beta_i}{q}.$$

Since we have

$$\sum_{i=1}^n \frac{\alpha_i}{p} + \frac{\beta_i}{q} = \sum_{i=1}^n \frac{\alpha_i}{p} + \sum_{i=1}^n \frac{\beta_i}{q} = \frac{1}{p} + \frac{1}{q} = 1,$$

we get

$$\sum_{i=1}^n a_i b_i \leq \left(\sum_{i=1}^n a_i^p\right)^{1/p} \left(\sum_{i=1}^n b_i^q\right)^{1/q}.$$

■

(G5.3) (Supplementary exercise)

Use the Hölder inequality to prove the **Minkowski inequality**:

$$\left(\sum_{i=1}^n (a_i + b_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^n a_i^p\right)^{1/p} + \left(\sum_{i=1}^n b_i^p\right)^{1/p},$$

for $p \in [1, \infty[$ and for $a_i, b_i \in [0, \infty[$ for $1 \leq i \leq n$.

Solution.

For $p = 1$ this is immediate, so assume $p > 1$. Let $q \in]1, \infty[$ satisfy

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and define $c_i := (a_i + b_i)^{p-1}$ for $1 \leq i \leq n$. Then $c_i^q = (a_i + b_i)^{(p-1)q} = (a_i + b_i)^p$, since $(p-1)q = p$. So

$$\left(\sum_{i=1}^n c_i^q\right)^{1/q} = \left(\sum_{i=1}^n (a_i + b_i)^p\right)^{1/q}.$$

We have

$$\sum_{i=1}^n c_i(a_i + b_i) = \sum_{i=1}^n c_i a_i + \sum_{i=1}^n c_i b_i,$$

and by Hölder's inequality we have

$$\sum_{i=1}^n c_i a_i + \sum_{i=1}^n c_i b_i \leq \left(\sum_{i=1}^n c_i^q\right)^{1/q} \left(\sum_{i=1}^n a_i^p\right)^{1/p} + \left(\sum_{i=1}^n c_i^q\right)^{1/q} \left(\sum_{i=1}^n b_i^p\right)^{1/p}.$$

Hence

$$\sum_{i=1}^n c_i(a_i + b_i) \leq \left(\left(\sum_{i=1}^n a_i^p\right)^{1/p} + \left(\sum_{i=1}^n b_i^p\right)^{1/p}\right) \left(\sum_{i=1}^n (a_i + b_i)^p\right)^{1/q}.$$

On the other hand

$$\sum_{i=1}^n c_i(a_i + b_i) = \sum_{i=1}^n (a_i + b_i)^p,$$

so in total we have

$$\frac{\sum_{i=1}^n (a_i + b_i)^p}{\left(\sum_{i=1}^n (a_i + b_i)^p\right)^{1/q}} \leq \left(\left(\sum_{i=1}^n a_i^p\right)^{1/p} + \left(\sum_{i=1}^n b_i^p\right)^{1/p}\right),$$

i.e.,

$$\left(\sum_{i=1}^n (a_i + b_i)^p\right)^{1/p} \leq \left(\left(\sum_{i=1}^n a_i^p\right)^{1/p} + \left(\sum_{i=1}^n b_i^p\right)^{1/p}\right),$$

since $1 - 1/q = 1/p$.

■