Fachbereich Mathematik
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4. Exercise sheet Analysis II for MCS
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Minitest

Let $I \subseteq \mathbb{R}$ be an interval, $f\colon I \to \mathbb{R}$ be a differentiable function and let x_0 be an interior point of I. Which of the following statements are correct?
(You should not spend more than 10 minutes on the test.)

- ☐ If f has a local minimum or maximum at x_0, then $f'(x_0) = 0$.
- ☐ If $f'(x_0) = 0$, then f has a local minimum or maximum at x_0.
- ☐ If $f'(x_0) = 0$ and $f''(x_0) \neq 0$, then f has a local minimum or maximum at x_0.
- ☐ If $f'(x_0) = 0$ and $f''(x_0) = 0$, then f does not have a local minimum or maximum at x_0.

Solution. The first statement is correct because of Corollary 4.27.

The second statement is false. Consider the following counterexample:

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3.$$

Then $f'(0) = 0$ but f does not have a local extremum at 0.

The third statement is correct due to Proposition 4.57 and its similar version for local maxima.

The fourth statement is false. Consider the counterexample:

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = x^4.$$

Then $f'(0) = f''(0) = 0$, but f has a local minimum at 0. ■

(G4.1)

(i) Determine the local maxima and minima of the function

$$f :]0,\infty[\to \mathbb{R}, \quad f(x) = x^x.$$

(ii) Determine any global maxima and minima of the function $f : [0, 2\pi] \to \mathbb{R}$ given by $f(x) = e^x \sin x$.

Recall that for a set X and a function $g : X \to \mathbb{R}$, g attains its global maximum in $a \in X$ if $g(a) = \max g(X)$. In the same way, g attains its global minimum in $a \in X$ if $g(a) = \min g(X)$.

Solution.

(i) Using $f(x) = x^x = \exp(x \log x)$, we obtain

$$\begin{aligned} f'(x) &= \exp(x\log x)(\log x + 1) = x^x(\log x + 1), \\ f''(x) &= x^x\left(\frac{1}{x} + (\log x + 1)^2\right). \end{aligned}$$

Since $x^x > 0$ for any $x > 0$,

$$f'(x) = 0 \quad\Leftrightarrow\quad x^x(\log x + 1) = 0 \quad\Leftrightarrow\quad \log x + 1 = 0 \quad\Leftrightarrow\quad x = \frac{1}{e}.$$

Thus, $x = \frac{1}{e}$ is the only stationary point of f. Since $f''(x) > 0$ for any $x \in]0,\infty[$, it follows that f has a local minimum at $x = \frac{1}{e}$. Since f is everywhere differentiable and since the domain of f is open we know that there are no other local minima or maxima.

(ii) By Theorem 3.52 in Hofmann we conclude that f attains its global maximum and its global minimum, since f is continuous and $[0, 2\pi]$ compact. Since f is differentiable the candidates are the endpoints of the domain and those x for which $f'(x) = 0$. We have

$$f'(x) = e^x \sin x + e^x \cos x,$$

so $f'(x) = 0$ if and only if $\sin x = -\cos x$. So the stationary points are $x_1 = \frac{3\pi}{4}$ and $x_2 = \frac{7\pi}{4}$. Since $f(0) = f(2\pi) = 0$ and

$$f(x_2) = \frac{-e^{\frac{7\pi}{4}}}{\sqrt{2}} < 0 < \frac{e^{\frac{3\pi}{4}}}{\sqrt{2}} = f(x_1),$$

we get that f attains its global maximum in $\frac{3\pi}{4}$ and its global minimum in $\frac{7\pi}{4}$. Nowhere else does f attain its global maximum or its global minimum.

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(G4.2) (Leibniz Rule)
Let $D \subseteq \mathbb{R}, n \in \mathbb{N}$, and $f, g\colon D \to \mathbb{R}$ be two n-times differentiable functions. Prove the Leibniz Rule:

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x).$$

Hint: Look at the proof of the binomial formula.

Solution. We prove the Leibniz Rule by mathematical induction:

$n = 1:$ The rule becomes the Product Rule:

$$\sum_{k=0}^{1} \binom{1}{k} f^{(1-k)}(x) g^{(k)}(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) = (f \cdot g)'(x).$$

$n \Rightarrow n+1:$

$$\begin{aligned}
(fg)^{(n+1)}(x) &= \left[(fg)^{(n)}\right]'(x) \\
&= \left[\sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}\right]'(x) \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k+1)}(x) g^{(n-k)}(x) + f^{(k)}(x) g^{(n-k+1)}(x)\right) \\
&= \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(k)}(x) g^{(n-k+1)}(x) + \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k+1)}(x) \\
&= f^{(n+1)}(x) g(x) + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k}\right) f^{(k)}(x) g^{(n-k+1)}(x) + f(x) g^{(n+1)}(x) \\
&= f^{(n+1)}(x) g(x) + \sum_{k=1}^{n} \binom{n+1}{k} f^{(k)}(x) g^{(n-k+1)}(x) + f(x) g^{(n+1)}(x) \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(x) g^{(n-k+1)}(x).
\end{aligned}$$

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