Fachbereich Mathematik
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3. Exercise sheet Analysis II for MCS
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(G3.1) Compute the limit

$$\lim_{\substack{x\to 0\\ x\neq 0}} \frac{\log(\cos x)}{x^2}.$$

Solution. Applying twice the Rule of Bernoulli and de l'Hôpital, we get that

$$\begin{aligned}\lim_{\substack{x\to 0\\ x\neq 0}} \frac{\log(\cos x)}{x^2} &= \lim_{\substack{x\to 0\\ x\neq 0}} \frac{-\frac{\sin x}{\cos x}}{2x} = \lim_{\substack{x\to 0\\ x\neq 0}} \frac{-\tan x}{2x} \\ &= \lim_{\substack{x\to 0\\ x\neq 0}} \frac{-\frac{1}{\cos^2 x}}{2} = -\frac{1}{2}.\end{aligned}$$

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(G3.2)

(i) Prove that for $x > 0$,

$$x^\alpha - \alpha x \leq 1 - \alpha \quad (0 < \alpha < 1). \tag{1}$$

(ii) Prove the arithmetical-geometrical inequality

$$a^\alpha b^\beta \leq \alpha a + \beta b \quad (a, b, \alpha, \beta > 0, \alpha + \beta = 1). \tag{2}$$

(iii) Generalize (2) to

$$\prod_{i=1}^{n} a_i^{\alpha_i} \leq \sum_{i=1}^{n} \alpha_i a_i \quad (n \in \mathbb{N}, a_i, \alpha_i > 0, \sum_{i=1}^{n} \alpha_i = 1). \tag{3}$$

(iv) Prove the arithmetic-geometric inequality

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} a_i \quad (a_i > 0, n \in \mathbb{N}). \tag{4}$$

Solution.

(i) Let us consider the function $f :]0,\infty[\to \mathbb{R}$, $f(x) = x^\alpha - \alpha x$. Then f is differentiable, and $f'(x) = \alpha x^{\alpha-1} - \alpha = \alpha(x^{\alpha-1} - 1)$. Since $\alpha \in]0,1[$, we have that $\alpha - 1 < 0$, so $f'(1) = 0$, $f'(x) < 0$ for $x > 1$, and $f'(x) > 0$ for $x < 1$. Thus, f is increasing on $]0,1]$, and decreasing on $[1,\infty[$. It follows that f attains a global maximum at $x = 1$. Since $f(1) = 1 - \alpha$, we get $f(x) \leq 1 - \alpha$ for any $x > 0$. That is, (1).

(ii) Apply (1) for $x = \frac{a}{b}$. We get

$$\frac{a^\alpha}{b^\alpha} - \frac{\alpha a}{b} \leq 1 - \alpha,$$

and so

$$\frac{a^\alpha}{b^\alpha} \leq \beta + \frac{\alpha a}{b}.$$

Hence

$$\frac{a^\alpha}{b^{1-\beta}} \leq \beta + \frac{\alpha a}{b},$$

that is,

$$\frac{b^\beta a^\alpha}{b} \leq \beta + \frac{\alpha a}{b}.$$

And therefore

$$b^\beta a^\alpha \leq b\beta + \alpha a.$$

(iii) Use induction on n.

$n = 1$: Obvious.

$n \Rightarrow n+1$:

Let $a_i, \alpha_i > 0$ for $1 \leq i \leq n+1$ such that $\sum_{i=1}^{n+1} \alpha_i = 1$. We have to prove that

$$\prod_{i=1}^{n+1} a_i^{\alpha_i} \leq \sum_{i=1}^{n+1} \alpha_i a_i.$$

Let $\beta := 1 - \alpha_{n+1} = \sum_{i=1}^{n} \alpha_i$. Then

$$\begin{aligned}
\prod_{i=1}^{n+1} a_i^{\alpha_i} &= \left(\prod_{i=1}^{n} a_i^{\frac{\alpha_i}{\beta}} \right)^{\beta} \cdot a_{n+1}^{\alpha_{n+1}} \\
&\leq \beta \left(\prod_{i=1}^{n} a_i^{\frac{\alpha_i}{\beta}} \right) + \alpha_{n+1} a_{n+1} \\
&\quad \text{(by (2) with } a := a_{n+1},\ b := \prod_{i=1}^{n} a_i^{\frac{\alpha_i}{\beta}},\ \alpha := \alpha_{n+1}) \\
&\leq \beta \sum_{i=1}^{n} \frac{\alpha_i}{\beta} a_i + \alpha_{n+1} a_{n+1} \\
&\quad \text{(by the induction hypothesis, since } \sum_{i=1}^{n} \frac{\alpha_i}{\beta} = 1) \\
&= \sum_{i=1}^{n+1} \alpha_i a_i.
\end{aligned}$$

(iv) Apply (3) for $\alpha_i = \frac{1}{n}$ for $1 \leq i \leq n$.

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