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2. Exercise sheet Analysis I for MCS
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(G2.1)

Let $I \subseteq \mathbb{R}$ be an open interval and let $a \in I$. Let $f : I \to \mathbb{R}$ be smooth, i.e. let f have derivatives $f^{(k)}$ for all $k \in \mathbb{N}$. Prove with the help of Taylor's formula that

$$f(a) = f'(a) = ... = f^{(n)}(a) = 0$$

implies

$$\lim_{x\to a} \frac{f(x)}{(x-a)^n} = 0.$$

Solution. Taylor's formula together with the hypothesis implies that for $x \in I$ $(x \neq a)$ there exists u located properly between a and x such that

$$f(x) = \frac{1}{(n+1)!} f^{(n+1)}(u)(x-a)^{(n+1)}.$$

Since $f^{(n+1)}$ is differentiable it is also continuous, and there exists a $\delta > 0$ such that $]a-\delta, a+\delta[\subseteq I$ and such that

$$|f^{(n+1)}(x) - f^{(n+1)}(a)| < 1$$

for $x \in]a-\delta, a+\delta[$. Thus for $x \in]a-\delta, a+\delta[$ we have

$$\left|\frac{f(x)}{(x-a)^n}\right| < \frac{1}{(n+1)!}(|f^{(n+1)}(a)|+1)|x-a|.$$

Hence

$$\lim_{x\to a}\left|\frac{f(x)}{(x-a)^n}\right| = 0.$$

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(G2.2)

Prove the following inequality with the aid of the Mean Value Theorem:

$$1 + \frac{x}{2\sqrt{1+x}} \leq \sqrt{1+x} \leq 1 + \frac{x}{2}, \quad -1 < x < \infty. \tag{1}$$

Solution. We consider the function $f\colon]-1,\infty[\to \mathbb{R}$ given by $f(x) = \sqrt{1+x}$. Then f is differentiable and $f'(x) = \frac{1}{2\sqrt{1+x}}$.
Let $x \in]-1,\infty[$. We have the following cases:

(i) $x \in]-1,0[$. We can apply the Mean Value Theorem to the restriction of f on $[x,0]$ to get $c \in]x,0[$ such that

$$\sqrt{1+x} - 1 = f(x) - f(0) = f'(c)(x-0) = \frac{1}{2\sqrt{1+c}}x.$$

Since $c \in]x,0[$ we get that $\sqrt{1+x} < \sqrt{1+c} < 1$, hence

$$\frac{1}{2} < \frac{1}{2\sqrt{1+c}} < \frac{1}{2\sqrt{1+x}}.$$

So since $x < 0$ we have

$$\frac{x}{2\sqrt{1+x}} < \frac{x}{2\sqrt{1+c}} < \frac{x}{2}.$$

Hence,

$$\frac{x}{2\sqrt{1+x}} < \sqrt{1+x} - 1 < \frac{x}{2},$$

which gives (1).

(ii) $x = 0$. Then we get equality in (1).

(iii) $x \in]0,\infty[$. The proof is similar to (a), by applying the Mean Value Theorem to the restriction of f on $[0,x]$.

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(G2.3) Supplementary exercise.

Use the previous exercise to conclude that

$$1 + \frac{x}{2+x} \leq \sqrt{1+x} \leq 1 + \frac{x}{2}, \quad 0 \leq x < \infty. \tag{2}$$

Use this inequality to estimate $\sqrt{102}$ with accuracy 10^{-3}.

Solution.

We prove $\frac{x}{2+x} \leq \frac{x}{2\sqrt{1+x}}$ and then apply (G2.2) to get (2).
Let $x \geq 0$. Then by (G2.2) we have that $\sqrt{1+x} \leq 1 + \frac{x}{2}$, so $2\sqrt{1+x} \leq 2+x$. Thus

$$\frac{1}{2+x} \leq \frac{1}{2\sqrt{1+x}}.$$

We multiply this inequality by x and use the fact that $x \geq 0$ to get

$$\frac{x}{2+x} \leq \frac{x}{2\sqrt{1+x}}.$$

Hence (2) is established. To estimate $\sqrt{102}$ we first notice that $\sqrt{102} = 10 \cdot \sqrt{1.02} = 10 \cdot \sqrt{1 + 0.02}$. If we multiply (2) by 10 and set $x = 0.02$, we obtain:

$$10.099 \leq \sqrt{102} \leq 10.1 \ .$$

Hence we have determined $\sqrt{102}$ with accuracy $10.1 - 10.099 = 0.001 = 10^{-3}$. ■