Fachbereich Mathematik
Prof. Dr. U. Kohlenbach
E. Briseid, Dr. L. Leuştean

TECHNISCHE UNIVERSITÄT DARMSTADT

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1. Exercise sheet Analysis I for MCS
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(G1.1) Show that the following functions are differentiable and determine their derivatives:

(i) $f_1 :]0,\infty[\to \mathbb{R},\ f_1(x) := x^{\frac{1}{x}}$, (ii) $f_2 : \mathbb{R} \to \mathbb{R},\ f_2(x) := \cos x \sin^2 x$

(iii) $f_3 : \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f_3(x) := xe^{\frac{1}{x}}$, (iv) $f_4 :]0,\infty[\to \mathbb{R},\ f_4(x) := \sqrt{x} + \dfrac{1}{\sqrt{x^3}}$

(v) $f_5 : \mathbb{R} \to \mathbb{R},\ f_5(x) := \ln(\exp(x))$.

Solution. The functions are differentiable, since they are compositions of differentiable functions.

(i) $f_1(x) = \exp\left(\frac{1}{x}\ln x\right)$, so

$$f_1'(x) = \left(\exp\left(\frac{1}{x}\ln x\right)\right)' = \exp\left(\frac{1}{x}\ln x\right)\left(\frac{1}{x}\ln x\right)' = x^{\frac{1}{x}}\left(-\frac{1}{x^2}\ln x + \frac{1}{x^2}\right) = \frac{x^{\frac{1}{x}}(1-\ln x)}{x^2}.$$

(ii) $f_2'(x) = \left(2\cos^2 x - \sin^2 x\right)\sin x$.

(iii) $f_3'(x) = e^{\frac{1}{x}}\left(1 - \frac{1}{x}\right)$.

(iv) $f_4'(x) = \frac{1}{2\sqrt{x}} - \frac{3}{2\sqrt{x^5}}$.

(v) $f_5(x) = \ln(\exp(x)) = x$, so $f_5'(x) = 1$. ■

(G1.2) Use Taylor's Theorem to estimate $\sqrt{26}$ with an error smaller than 10^{-4}.

Solution. We let $f :]0,\infty[\to \mathbb{R}$ be defined by $f(x) := \sqrt{x}$. By differentiating we obtain $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, and $f'''(x) = \frac{3}{8}x^{-5/2}$. Now Taylor's Theorem (via Corollary 4.62) implies the existence of a number $u \in]25,26[$ such that

$$\sqrt{26} = f(26) = f(25) + f'(25)(26-25) + \frac{f''(25)}{2}(26-25)^2 + \frac{f'''(u)}{6}(26-25)^3.$$

Since $25 < u < 26$, we have

$$|f'''(u)| < \frac{3}{8}\cdot 25^{-5/2} = \frac{3}{8}\cdot\frac{1}{3125} = \frac{3}{25000}.$$

Therefore

$$|\frac{f'''(u)}{6}(26-25)^3| < \frac{3}{6\cdot 25000} = \frac{1}{5}\cdot 10^{-4}.$$

And since

$$f(25) + f'(25)(26-25) + \frac{f''(25)}{2}(26-25)^2 = 5 + \frac{1}{2}\cdot 25^{-1/2} - \frac{1}{8}\cdot 25^{-3/2} = 5.09900,$$

we get

$$|\sqrt{26} - 5.09900| < \frac{1}{5}\cdot 10^{-4}.$$

In this exercise we considered the Taylor polynomial of degree 2 for f, since the error term of form

$$\frac{1}{n!}f^{(n)}(u)(x-a)^n$$

from Corollary 4.62 would have been too large if we had used the Taylor polynomial of degree 1 for f at 25. ■