Solutions (sketches)

1. (i) Case 1: $(\forall y \in X) P(y)$. Pick for x an arbitrary element of X.

   Case 2: $(\exists x \in X) \neg P(x)$. Take such an x.

(ii) Assume not, that is

$(\exists n_1)(\forall m > n_1)(f(m) \neq 0)$ and

$(\exists n_2)(\forall m > n_2)(f(m) \neq 1)$.

Take $n_3 := \max\{n_1, n_2\} + 1$. Then $0 \neq f(n_3) \neq 1$, a contradiction.

(iii) $n = 1$ ✓

$n \mapsto n+1$: $\displaystyle\sum_{i=1}^{n+1} i^2 \overset{\text{I.H.}}{=} \frac{n(n+1)(2n+1)}{6} + (n+1)^2 =$

$$= \frac{(n+1)}{6}(2n^2 + n + 6n + 6) = \frac{(n+1)(n+2)(2n+3)}{6}.$$

$\square$

2. (i) The reflexivity, symmetry and transitivity of $\sim$ follow from the corresponding properties of $=$.

Let $[x]$ be the $\sim$-equivalence class of x.

$Y := \{[x] : x \in X\}$, $f : X \to Y$, $f(x) = [x]$.

$x_1 \sim x_2 \iff f(x_1) = f(x_2)$.

(ii) $d(f,g) \le d(f,u) + d(u,v) + d(g,v)$. Hence

a) $d(f,g) - d(u,v) \le d(f,u) + d(g,v)$.

$d(u,v) \le d(u,f) + d(f,g) + d(g,v)$. Hence

b) $d(u,v) - d(f,g) \le d(f,u) + d(g,v)$.

a) + b) $\Rightarrow$ claim. $\square$

3. (i) Since $\frac{1}{2^k}\frac{|a_k|}{1+|a_k|} \le \frac{1}{2^k}$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges, the series converges by the comparison criterion.

The estimate follows (using the hint) from

$$\frac{|a+b|}{1+|a+b|} \le \frac{|a|+|b|}{1+|a|+|b|} = \frac{|a|}{1+|a|+|b|} + \frac{|b|}{1+|a|+|b|} \le$$

$$\le \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}.$$

(ii) $x_n = \begin{cases} x & \text{if } n \text{ is even} \\ 1-x & \text{if } n \text{ is odd}. \end{cases}$

4. (i) For $x \neq 0$ both functions are continuous in x since $\sin$, $\cos$, $\frac{1}{x}$ are continuous.

$x = 0$: $\lim\limits_{\substack{x \to 0 \\ x \neq 0}} f(x) = 0 = f(0)$ since $\lim\limits_{\substack{x \to 0 \\ x \neq 0}} \sin x = 0$ and cos is bounded. Hence f is also continuous at 0.

To show that g is discontinuous at $x=0$ take

$x_n := \frac{1}{\frac{\pi}{2}+2n\pi} \xrightarrow{n\to\infty} 0$. Then $\sin\frac{1}{x_n} = \sin\left(\frac{\pi}{2}+2n\pi\right)=1$

and $\cos x_n \xrightarrow{n\to\infty} 1$. Thus, $g(x_n) \xrightarrow{n\to\infty} 1$ whereas $g(0)=0$.

(ii) Assume not, i.e. $(\exists \varepsilon>0)\,(\forall n)\,(\exists x_n, y_n \in [0,1])$

with $x_n+\varepsilon \le y_n$ and $f(x_n)+\frac{1}{n} > f(y_n)$.

By the Bolzano-Weierstraß Theorem, $(x_n), (y_n)$ have

convergent subsequences $x_{n_k} \to x$, $y_{n_k} \to y$.

Clearly $x+\varepsilon \le y$ but (by the continuity of f in x, y)

$f(x) \geq f(y)$ which contradicts the strict monotonicity of f. □

5. (i) $f'(x) = (5x^4+3)\cos x$

$g(x) = 2x$, so $g'(x)=2$.

(ii) Let $c\in\mathbb{R}$ and assume that there are $x_1 \neq x_2 \in [a,b]$

s.t. $f(x_1)=f(x_2)=c$. By Rolle's Theorem we get x_0

between x_1 and x_2 s.t. $f'(x_0)=0$. This contradicts the

hypothesis.

(iii) Let $g: \mathbb{R}\to\mathbb{R}$, $g(x) = \frac{f(x)}{e^x}$. Then $g'(x)=0$ for all

$x\in\mathbb{R}$, so $g(x) = C$ for some constant C. Hence, $f(x) = C\cdot e^x$. □

6. (i) $f'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2)$

$f''(x) = 12x - 18$

$f'(x) = 0 \iff x = 1$ or $x = 2$.

$f''(1) = -6 < 0 \Rightarrow f$ has a local maximum at $x=1$

$f''(2) = 6 > 0 \Rightarrow f$ has a local minimum at $x=2$.

(ii) f is continuous on the compact interval $[0,3]$, so f has global extrema. The points at which these global extrema are attained are between the endpoints 0,3 and the stationary points 1,2.

$$\left.\begin{array}{l} f(0) = -5 \\ f(1) = 0 \\ f(2) = -1 \\ f(3) = 4 \end{array}\right\} \Rightarrow f \text{ attains a global minimum at } x=0 \text{ and a global maximum at } x=3.$$

(iii) By Taylor's Theorem we get that

$$f(x) = \sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!} \cdot x^k + \frac{f^{(4)}(u)}{4!} \cdot x^4 \quad \text{for } u \in]0,x[$$

$$= \sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!} \cdot x^k \quad \text{since } f^{(4)} = 0.$$

7. (i) (a) $\int_0^1 x e^x \, dx = \int_0^1 x \cdot (e^x)' \, dx = x e^x \Big|_0^1 - \int_0^1 e^x \, dx =$

$$= x e^x \Big|_0^1 - e^x \Big|_0^1 = e - (e-1) = 1.$$

(b) $\int_0^2 |x-1| \, dx = \int_0^1 (1-x)\,dx + \int_1^2 (x-1)\,dx =$

$$= \left(x - \frac{x^2}{2}\right)\Big|_0^1 - \left(\frac{x^2}{2} - x\right)\Big|_1^2 = \frac{1}{2} + \frac{1}{2} = 1.$$

(ii) Let $g : [a,b] \to \mathbb{K}$, $g(x) = f(x) - x$. Then $\int_a^b g(x)\,dx = 0$.

It follows that there exists $x_0 \in [a,b]$ s.t. $g(x_0) = 0$ (see Analysis II, G6.2). That is, $f(x_0) = x_0$.

8. (i) Verify the axioms (N1)-(N3)

(N1) $\|x\| \geq 0$

$\|x\| = 0 \iff \|x\|_V + \|T(x)\|_W = 0 \iff \|x\|_V = 0$ and $\|T(x)\|_W = 0$

$\iff x = 0.$

(N2) For $\lambda \in \mathbb{K}, x \in V$,

$\|\lambda x\| = \|\lambda x\|_V + \|T(\lambda x)\|_W = |\lambda| \cdot \|x\|_V + |\lambda| \cdot \|T(x)\|_W$

$= |\lambda| \cdot \|x\|_V$

(N3) For $x, y \in V$,

$\|x+y\| = \|x+y\|_V + \|T(x+y)\|_W = \|x+y\|_V + \|T(x) + T(y)\|_W$

$\leq \|x\|_V + \|y\|_V + \|T(x)\|_W + \|T(y)\|_W = \|x\| + \|y\|$

(ii) Let $(y_n)_n$ be a sequence in tA s.t. $\lim_{n\to\infty} y_n = y$.

$y_n = t x_n$ with $(x_n)_n$ a sequence in A.

Then $\lim_{n\to\infty} x_n = \frac{1}{t} y$ and, since A is closed, $\frac{1}{t} y \in A$.

It follows that $y = t \cdot \left(\frac{1}{t} y \right) \in tA$.

□

9. (i)

$$\frac{\partial f}{\partial x_i}(x) = 2x_i \quad \text{for all } i = 1, \dots, n, \ x = (x_1 \dots x_n) \in \mathbb{R}^n.$$

Since all partial derivatives are continuous, f is differentiable

and for all $x \in \mathbb{R}^n, v \in \mathbb{R}^n$

$$f'(x)(v) = \partial f(x) \cdot v = (2x_1 \dots 2x_n) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = 2 \sum_{i=1}^{n} x_i v_i$$

(ii) By differentiating with respect to t (using the Chain Rule),

we get

$$f(x) = f'(tx)(x)$$

Put $t = 0$.

□

10. (i) $$\partial f(x,y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

Since f has continuous partial derivatives, it follows that f is continuously differentiable and for all $(x,y) \in \mathbb{R}^2$, $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathbb{R}^2$,

$$f'(x,y)(h) = \partial f(x,y) \cdot h = \begin{pmatrix} 2xh_1 - 2yh_2 \\ 2yh_1 + 2xh_2 \end{pmatrix}.$$

(ii) For $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $\det \partial f(x,y) = 4x^2 + 4y^2 \neq 0$,

so $f'(x,y)$ is invertible.

We can apply the Inverse Function theorem to get that f is locally invertible.

(iii) f is not globally invertible, since it is not injective.

Indeed, $f(1,1) = f(-1,1)$.