TECHNISCHE
UNIVERSITÄT
DARMSTADT
Dr. L. Leuştean
September 5, 2006

Fachbereich Mathematik

MCS Analysis II

Please write your name on each sheet and number all pages. At the end of the examination, put the sheets with your solutions in the folded examination sheet.

Name: ....
First name: ....
Matr.-Nr.: ....
Studies: ....
☐ Studienleisteung (Schein)
☐ Prüfungsleistung (Bachelor)

Important:

- Time for the examination: **120 Minutes**. Total amount of points: **50 points**.
- **Admitted material:** 2 handwritten A4 sheets with your signature.
- All steps in the solutions and partial results need sufficient explanation.
- **Good luck!**

Problem	1	2	3	4	5	Σ	Note
Maximal Points	10	10	10	10	10	50	
Given Points							

Problem 1: Extrema and Taylor series **(10 points)**

Consider the function $f:[0,3]\to\mathbb{R}, \quad f(x)=2x^3-9x^2+12x-5.$

(i) (4 points) Determine the local minima and maxima of f.

(ii) (3 points) Prove that f has global extrema and determine them.

(iii) (3 points) Prove that for any $x\in[0,3]$,

$$f(x)=f(0)+\sum_{k=1}^{3}\frac{f^{(k)}(0)}{k!}x^k.$$

Problem 2: Riemann Integral **(10 points)**

(i) (6 points) Compute the following Riemann integrals:

(a) $\displaystyle\int_0^1 x\,e^x dx;$ (b) $\displaystyle\int_0^2 |x-1|dx.$

(ii) (4 points) Let $f:[a,b]\to\mathbb{R}$ be a continuous function such that $\int_a^b f(x)dx = \frac{1}{2}(b^2-a^2)$. Show that there exists $x_0\in[a,b]$ such that $f(x_0)=x_0$.

Problem 3: Normed spaces **(10 points)**

(i) (5 points) Suppose that $(V,\|\cdot\|_V)$ and $(W,\|\cdot\|_W)$ are normed spaces over $\mathbb{K}$, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. Let $T:V\to W$ be a linear transformation and define the function $\|\cdot\|:V\to\mathbb{R}$ by

$$\|x\| := \|x\|_V + \|T(x)\|_W \qquad \text{for all } x\in V.$$

Prove that $\|\cdot\|$ is a norm on V.

(ii) (5 points) Let V be a normed space over $\mathbb{R}$, $A\subseteq V$ be closed in V and $t\in\mathbb{R}, t>0$. Define $tA := \{tx \mid x\in A\}$. Prove that tA is closed in V.

Problem 4: Differentiability in $\mathbb{R}^n$ **(10 points)**

(i) (5 points) Let $f:\mathbb{R}^n\to\mathbb{R}$, $f(x_1,\dots,x_n)=x_1^2+x_2^2+\ldots+x_n^2$. Show that f is differentiable on $\mathbb{R}^n$ and compute its derivative.

(ii) (5 points) Let $g:\mathbb{R}^n\to\mathbb{R}$ be a differentiable function such that $g(tx)=tg(x)$ for all $t\in\mathbb{R}$ and all $x\in\mathbb{R}^n$. Show that $g(x)=g'(0)(x)$ for all $x\in\mathbb{R}^n$.

Problem 5: Inverse Function Theorem **(10 points)**

Consider the function

$$f:\mathbb{R}^2\to\mathbb{R}^2, \qquad f(x,y)=\begin{pmatrix} x^2-y^2 \\ 2xy \end{pmatrix}.$$

(i) (4 points) Show that f is continuously differentiable and compute its derivative.

(ii) (3 points) Show that f is locally invertible around every point $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$.

(iii) (3 points) Does f have a global inverse?