



## MCS Analysis I

Please write your name on each sheet and number all pages. At the end of the examination, put the sheets with your solutions in the folded examination sheet.

Name: .....

First name: .....

Matr.-Nr.: .....

Studies: .....

Studienleistung (Schein)

Prüfungsleistung (Bachelor)

### Important:

- Time for the examination: **120 Minutes**. Total amount of points: **50 points**.
- **Admitted material:** 2 handwritten A4 sheets with your signature.
- All steps in the solutions and partial results need sufficient explanation.
- **Good luck!**

Problem	1	2	3	4	5	$\Sigma$	Note
Maximal Points	10	10	10	10	10	50	
Given Points							

### Problem 1: Logic and Natural Numbers

(10 points)

- (i) (3 points) Show that the following sentence holds for all nonempty sets  $X \neq \emptyset$  and all unary predicates  $P$  on  $X$ :

$$(\exists x \in X)[P(x) \rightarrow (\forall y \in X)P(y)].$$

- (ii) (3 points) Let  $f: \mathbb{N} \rightarrow \{0, 1\}$ . Show that

$$(\forall n)(\exists m > n)(f(m) = 0) \vee (\forall n)(\exists m > n)(f(m) = 1).$$

- (iii) (4 points) Prove by induction that  $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$ .

### Problem 2: Functions and Metric Spaces

(10 points)

- (i) (5 points) Let  $X, Y$  be nonempty sets and  $f: X \rightarrow Y$  be an arbitrary function. Define  $x_1 \sim x_2 : \iff f(x_1) = f(x_2)$ . Show that  $\sim$  is an equivalence relation on  $X$  and that every equivalence relation on  $X$  can be obtained in this way (for suitable  $Y$  and  $f$ ).

(ii) (5 points) Let  $(X, d)$  be a metric space. Prove that

$$|d(f, g) - d(u, v)| \leq d(f, u) + d(g, v)$$

for all  $f, g, u, v \in X$ .

### Problem 3: Sequences and series

(10 points)

(i) (5 points) Show that for each sequence  $(a_n)_n$  in  $\mathbb{R}$

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|a_k|}{1+|a_k|} \quad \text{converges}$$

and that for all sequences  $(a_n)_n, (b_n)_n$

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|a_k + b_k|}{1+|a_k + b_k|} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|a_k|}{1+|a_k|} + \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|b_k|}{1+|b_k|}$$

Hint:  $t \mapsto \frac{t}{1+t}$  is increasing for  $t > -1$ .

(ii) (5 points) Let  $f: [0, 1] \rightarrow [0, 1]$ ,  $f(x) = 1 - x$ . For  $x \in [0, 1]$  define  $x_0 := x, x_{n+1} := f(x_n)$ . For which  $x \in [0, 1]$  does  $(x_n)_n$  converge?

### Problem 4: Continuity

(10 points)

(i) (5 points) Determine in which points  $x \in \mathbb{R}$  the functions

$$f: \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = \begin{cases} (\sin x) \left(\cos \frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and

$$g: \mathbb{R} \rightarrow \mathbb{R}, \quad g(x) = \begin{cases} (\cos x) \left(\sin \frac{1}{x}\right) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

are continuous?

(ii) (5 points) Let  $f: [0, 1] \rightarrow \mathbb{R}$  be continuous and strictly increasing. Show that  $f$  is uniformly strictly increasing in the sense

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x_1, x_2 \in [0, 1])(x_1 + \varepsilon \leq x_2 \Rightarrow f(x_1) + \delta \leq f(x_2)).$$

### Problem 5: Differentiability in $\mathbb{R}$

(10 points)

(i) (4 points) Compute the derivatives of the following functions:

$$(a) f: \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = \sin(x^5 + 3x); \quad (b) g: ]0, \infty[ \rightarrow \mathbb{R}, \quad g(x) = 2 \exp(\ln x).$$

(ii) (3 points) Let  $f: [a, b] \rightarrow \mathbb{R}$  be a differentiable function such that  $f'(x) > 0$  for all  $x \in [a, b]$ . Show that for any  $c \in \mathbb{R}$ , the equation  $f(x) = c$  has at most one solution.

(iii) (3 points) Determine all differentiable functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  satisfying

$$(\forall x \in \mathbb{R})(f'(x) = f(x)).$$