TECHNISCHE
UNIVERSITÄT
DARMSTADT
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Fachbereich Mathematik

Basismodul Analysis f. MCS-BSc

Please write your name on each sheet and number all pages. At the end of the examination, put the sheets with your solutions in the folded examination sheet.

Name: ....
First name: ....
Matr.-Nr.: ....
Studies: ....
☐ Studienleisteung (Schein)
☐ Prüfungsleistung (Bachelor)

Important:

- Time for the examination: **240 Minutes**. Total amount of points: **100 points**.
- **Admitted material:** 4 handwritten A4 sheets with your signature.
- All steps in the solutions and partial results need sufficient explanation.
- **Good luck!**

Problem	1	2	3	4	5	6	7	8	9	10	Σ	Note
Maximal Points	10	10	10	10	10	10	10	10	10	10	100	
Given Points												

Analysis I

Problem 1: Logic and Natural Numbers (10 points)

(i) (3 points) Show that the following sentence holds for all nonempty sets $X \neq \emptyset$ and all unary predicates P on X:

$$(\exists x \in X)[P(x) \to (\forall y \in X)P(y)].$$

(ii) (3 points) Let $f : \mathbb{N} \to \{0,1\}$. Show that

$$(\forall n)(\exists m > n)(f(m) = 0) \vee (\forall n)(\exists m > n)(f(m) = 1).$$

(iii) (4 points) Prove by induction that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Problem 2: Functions and Metric Spaces **(10 points)**

(i) (5 points) Let X, Y be nonempty sets and $f: X \to Y$ be an arbitrary function. Define $x_1 \sim x_2 :\Longleftrightarrow f(x_1) = f(x_2)$. Show that $\sim$ is an equivalence relation on X and that every equivalence relation on X can be obtained in this way (for suitable Y and f).

(ii) (5 points) Let (X,d) be a metric space. Prove that

$$|d(f,g) - d(u,v)| \le d(f,u) + d(g,v)$$

for all $f,g,u,v \in X$.

Problem 3: Sequences and series **(10 points)**

(i) (5 points) Show that for each sequence $(a_n)_n$ in $\mathbb{R}$

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|a_k|}{1+|a_k|} \quad \text{converges}$$

and that for all sequences $(a_n)_n, (b_n)_n$

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|a_k+b_k|}{1+|a_k+b_k|} \le \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|a_k|}{1+|a_k|} + \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|b_k|}{1+|b_k|}$$

Hint: $t \mapsto \frac{t}{1+t}$ is increasing for $t > -1$.

(ii) (5 points) Let $f: [0,1] \to [0,1]$, $f(x) = 1 - x$. For $x \in [0,1]$ define $x_0 := x, x_{n+1} := f(x_n)$. For which $x \in [0,1]$ does $(x_n)_n$ converge?

Problem 4: Continuity **(10 points)**

(i) (5 points) Determine in which points $x \in \mathbb{R}$ the functions

$$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} (\sin x)\left(\cos \frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and

$$g: \mathbb{R} \to \mathbb{R}, \quad g(x) = \begin{cases} (\cos x)\left(\sin \frac{1}{x}\right) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

are continuous?

(ii) (5 points) Let $f: [0,1] \to \mathbb{R}$ be continuous and strictly increasing. Show that f is uniformly strictly increasing in the sense

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x_1, x_2 \in [0,1])\big(x_1 + \varepsilon \le x_2 \Rightarrow f(x_1) + \delta \le f(x_2)\big).$$

Problem 5: Differentiability in $\mathbb{R}$ **(10 points)**

(i) (4 points) Compute the derivatives of the following functions:

(a) $f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \sin(x^5 + 3x)$; (b) $g:]0,\infty[\to \mathbb{R}, \quad g(x) = 2\exp(\ln x)$.

(ii) (3 points) Let $f: [a,b] \to \mathbb{R}$ be a differentiable function such that $f'(x) > 0$ for all $x \in [a,b]$. Show that for any $c \in \mathbb{R}$, the equation $f(x) = c$ has at most one solution.

(iii) (3 points) Determine all differentiable functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$(\forall x\in\mathbb{R})(f'(x)=f(x)).$$

Analysis II

Problem 6: Extrema and Taylor series **(10 points)**

Consider the function $f:[0,3]\to\mathbb{R},\quad f(x)=2x^3-9x^2+12x-5.$

(i) (4 points) Determine the local minima and maxima of f.

(ii) (3 points) Prove that f has global extrema and determine them.

(iii) (3 points) Prove that for any $x\in[0,3]$,

$$f(x)=f(0)+\sum_{k=1}^{3}\frac{f^{(k)}(0)}{k!}x^k.$$

Problem 7: Riemann Integral **(10 points)**

(i) (6 points) Compute the following Riemann integrals:

(a) $\displaystyle\int_0^1 x\,e^x\,dx;$ (b) $\displaystyle\int_0^2 |x-1|\,dx.$

(ii) (4 points) Let $f:[a,b]\to\mathbb{R}$ be a continuous function such that $\displaystyle\int_a^b f(x)dx=\frac{1}{2}(b^2-a^2)$. Show that there exists $x_0\in[a,b]$ such that $f(x_0)=x_0$.

Problem 8: Normed spaces **(10 points)**

(i) (5 points) Suppose that $(V,\|\cdot\|_V)$ and $(W,\|\cdot\|_W)$ are normed spaces over $\mathbb{K}$, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. Let $T:V\to W$ be a linear transformation and define the function $\|\cdot\|:V\to\mathbb{R}$ by

$$\|x\|:=\|x\|_V+\|T(x)\|_W \qquad \text{for all } x\in V.$$

Prove that $\|\cdot\|$ is a norm on V.

(ii) (5 points) Let V be a normed space over $\mathbb{R}$, $A\subseteq V$ be closed in V and $t\in\mathbb{R}, t>0$. Define $tA:=\{tx\mid x\in A\}$. Prove that tA is closed in V.

Problem 9: Differentiability in $\mathbb{R}^n$ **(10 points)**

(i) (5 points) Let $f:\mathbb{R}^n\to\mathbb{R},\ f(x_1,\dots,x_n)=x_1^2+x_2^2+\ldots+x_n^2$. Show that f is differentiable on $\mathbb{R}^n$ and compute its derivative.

(ii) (5 points) Let $g:\mathbb{R}^n\to\mathbb{R}$ be a differentiable function such that $g(tx)=tg(x)$ for all $t\in\mathbb{R}$ and all $x\in\mathbb{R}^n$. Show that $g(x)=g'(0)(x)$ for all $x\in\mathbb{R}^n$.

Problem 10: Inverse Function Theorem **(10 points)**

Consider the function

$$f:\mathbb{R}^2\to\mathbb{R}^2,\qquad f(x,y)=\begin{pmatrix} x^2-y^2\\ 2xy\end{pmatrix}.$$

(i) (4 points) Show that f is continuously differentiable and compute its derivative.

(ii) (3 points) Show that f is locally invertible around every point $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

(iii) (3 points) Does f have a global inverse?