Linear transformations between normed spaces

In the sequel, V and W are normed spaces over $\mathbb{K}$. The norms in these spaces will be both denoted by $\|\cdot\|$, except when there is a strong reason for differentiating them.

Recall that by a <u>linear transformation</u> from V to W we mean a mapping $T: V \to W$ satisfying the following conditions:

(LT1) $(\forall x, y \in V)\,(T(x+y) = T(x) + T(y))$.

(LT2) $(\forall x \in V)(\forall \lambda \in \mathbb{K})\,(T(\lambda x) = \lambda \cdot T(x))$.

It is easy to see that if $T: V \to W$ is a linear transformation, then $T(0) = 0$.

Theorem 6.23

For any linear transformation $T: V \to W$, the following statements are equivalent:

(i) T is continuous.

(ii) T is continuous at $x=0$.

(iii) there exists $M>0$ s.t. $(\forall x \in V)\,(\|T(x)\| \leq M\|x\|)$.

Proof

(i) $\Rightarrow$ (ii) Trivial.

(ii) $\Rightarrow$ (iii) Assume that T is continuous at $x=0$. Then there exists $\delta>0$ s.t. $(\forall x \in V)\,(\|x-0\| < \delta \Rightarrow \|T(x) - T(0)\| \leq 1)$, that is

(1) $\quad (\forall x \in V)\,(\|x\| < \delta \Rightarrow \|T(x)\| \leq 1)$.

Let $x \in V$, $x \neq 0$. Then $\left\| \frac{\delta}{2\|x\|} \cdot x \right\| = \frac{\delta}{2} < \delta$ so, by (1),

$\left\|T\left(\frac{\delta}{2\|x\|}\cdot x\right)\right\| \leq 1$. On the other hand,

$$\left\|T\left(\frac{\delta}{2\|x\|}\cdot x\right)\right\| = \left\|\frac{\delta}{2\|x\|}\cdot T(x)\right\| = \frac{\delta}{2\|x\|}\cdot \|T(x)\|.$$

Thus, $\frac{\delta}{2\|x\|}\cdot \|T(x)\| \leq 1$, hence $\|T(x)\| \leq \frac{2}{\delta}\|x\|$.

Let $M := \frac{2}{\delta}$. Then $\|T(x)\| \leq M\cdot\|x\|$ for all $x \in V$, $x \neq 0$.

For $x = 0$, we have that $\|T(0)\| = 0 \leq M\cdot\|0\|$.

(iii) $\Rightarrow$ (i) Assume that $M > 0$ is such that $\|T(x)\| \leq M\cdot\|x\|$ for all $x \in V$. For distinct $x, y \in V$ we get that

$$\left\|\frac{x-y}{\|x-y\|}\right\| = 1 \text{ and } \left\|T\left(\frac{x-y}{\|x-y\|}\right)\right\| = \frac{1}{\|x-y\|}\cdot\|T(x-y)\| = \frac{\|T(x)-T(y)\|}{\|x-y\|}.$$

Thus, $\frac{\|T(x)-T(y)\|}{\|x-y\|} \leq M$, so

$$\|T(x)-T(y)\| \leq M\cdot\|x-y\|.$$

It follows that T is Lipschitz continuous and, as a consequence, T is continuous. $\square$

<u>Corollary 6.24</u>

Any continuous linear transformation $T: V \to W$ is bounded on the closed unit ball $B_1(0)$.

<u>Proof.</u>

For all $x \in B_1(0)$ we get that

$$\|T(x)\| \leq M\cdot\|x\| \leq M, \text{ since } \|x\| \leq 1.$$

$\square$

Notation 6.25

We denote by $\mathcal{L}(V,W)$ the set of continuous linear transformations from V to W.

Theorem 6.26

Let V and W be normed spaces over $\mathbb{K}$. The following assertions hold:

(i) $\mathcal{L}(V,W)$ is a vector space over $\mathbb{K}$ with respect to pointwise vector addition and scalar multiplication.

(ii) The function $\|\cdot\|: \mathcal{L}(V,W) \to \mathbb{R}$ defined for every $T \in \mathcal{L}(V,W)$ by

$$\|T\| \overset{\text{def}}{=} \sup\{\|T(x)\| : x \in V,\ \|x\| \leq 1\}$$

is a norm on $\mathcal{L}(V,W)$. Moreover, $\|T(x)\| \leq \|T\| \cdot \|x\|$ for all $x \in V$.

(iii) If W is a Banach space, then $\mathcal{L}(V,W)$ is also a Banach space.

Proof

(i), (ii): Easy exercises.

(iii) will be proved in the tutorials.

□

Definition 6.27

The norm on $\mathcal{L}(V,W)$ defined in the above theorem is called the operator norm.

Proposition 6.28

Let $T \in \mathcal{L}(V,W)$. Then

$$\|T\| = \sup\{\|T(x)\| : x \in V,\ \|x\| = 1\} = \inf\{M > 0 : \|T(x)\| \leq M \cdot \|x\| \text{ for all } x \in V\}$$

Proof. Exercise □

Proposition 6.28

Let V, W and U be normed spaces over $\mathbb{K}$. Then for every $T \in \mathcal{L}(V,W)$ and $S \in \mathcal{L}(W,U)$, we have $S \circ T \in \mathcal{L}(W,U)$ and

$$\|S\circ T\| \leq \|S\|\cdot\|T\|.$$

Proof.

It is easy to check that $S\circ T \in \mathcal{L}(W,U)$. On the other hand, we have that for all $x \in V$, $\|x\| \leq 1$,

$$\|(S\circ T)(x)\| = \|S(T(x))\| \leq \|S\|\cdot\|T(x)\| \leq \|S\|\cdot\|T\|\cdot\|x\| \leq \|S\|\cdot\|T\|.$$

Then,

$$\|S\circ T\| = \sup\{\|(S\circ T)(x)\| : x \in V, \|x\| \leq 1\} \leq \|S\|\cdot\|T\|.$$ □

Definition 6.30

Let V be a normed space over $\mathbb{K}$. A linear transformation $T: V \to \mathbb{K}$ is called a <u>linear functional on V.</u>

Theorem 6.31

Let V be a normed space over $\mathbb{K}$. The set V^* of all continuous linear functionals on V is a Banach space over $\mathbb{K}$ with the operator norm.

Proof.

$V^* = \mathcal{L}(V, \mathbb{K})$. Since $(\mathbb{K}, |\cdot|)$ is a Banach space, we can apply Theorem 6.26 to get that V^* is a Banach space with the operator norm

$$\|T\| = \sup\{|T(x)| : x \in V, \|x\| \leq 1\}.$$ □

Remark 6.32

The normed space V^* is called the dual space of V. Note that V^* is a Banach space irrespective of wether V is or not.

Definition 6.33

Let V and W be normed spaces over $\mathbb{K}$. A (topological) isomorphism of V onto W is a linear isomorphism $T: V \to W$ such that T and T^{-1} are continuous.

The spaces V and W are (topologically) isomorphic if there is a topological isomorphism between them.

Remark 6.34

The composition of two topological isomorphisms is again a topological isomorphism. Consequently, the relation of ~~being~~ topologically isomorphic is an equivalence on the class of normed spaces over $\mathbb{K}$.

Proposition 6.35

Let $T: V \to W$ be a surjective linear transformation. The following statements are equivalent:

(i) T is a topological isomorphism of normed spaces.

(ii) there exist $\alpha, \beta > 0$ such that

$$\alpha \|x\|_V \leq \|T(x)\|_W \leq \beta \|x\|_V \quad \text{for all } x \in V.$$

Proof

(i) $\Rightarrow$ (ii) Assume that T is an isomorphism. Then $T, T^{-1} \in \mathcal{L}(V, W)$ and, since they are bijections, $T, T^{-1} \neq 0$, so $\|T\|, \|T^{-1}\| \neq 0$.

By Theorem 6.26(ii), we have that $\|T(x)\| \le \|T\| \cdot \|x\|$ for all $x \in V$. Thus, we can take $\beta := \|T\| > 0$. Applying Theorem 6.26 (ii) for T^{-1} we obtain that $\|x\| = \|T^{-1}(T(x))\| \le \|T^{-1}\| \cdot \|T(x)\|$ for all $x \in V$. Then, taking $\alpha := \frac{1}{\|T^{-1}\|} > 0$, it follows that

$$\alpha \cdot \|x\| = \frac{1}{\|T^{-1}\|} \cdot \|x\| \le \|T(x)\| \quad \text{for all } x \in V.$$

(ii) $\Rightarrow$ (i) Assume now that (ii) holds. From theorem 6.23 and the fact that $\|T(x)\| \le \beta \cdot \|x\|$ for all $x \in V$, it follows that T is continuous. It is easy to see that T is injective:

$$T(x) = 0 \Rightarrow \|T(x)\| = 0 \Rightarrow \alpha \cdot \|x\| = 0 \Rightarrow \|x\| = 0 \Rightarrow x = 0, \text{ since } \alpha > 0.$$

Since T is surjective by the hypothesis, it follows that T is a linear isomorphism.

It remains to prove that T^{-1} is continuous. Let $y \in W$. Then there is $x \in V$ s.t. $T(x) = y$, since T is surjective. It follows that

$$\|T^{-1}(y)\| = \|T^{-1}(T(x))\| = \|x\| \le \frac{1}{\alpha} \cdot \|T(x)\| = \frac{1}{\alpha} \cdot \|y\|.$$

Applying again Theorem 6.23 to get that T^{-1} is continuous. $\square$

<u>Corollary 6.36</u>

Let V be a $\mathbb{K}$-vector space and $\|\cdot\|_1, \|\cdot\|_2$ be two norms on V.

Then

$\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent $\iff$ $1_V : (V, \|\cdot\|_1) \to (V, \|\cdot\|_2)$ is an isomorphism.

<u>Proof</u>

Apply Definition 6.18 and Proposition 6.35. $\square$

Finite dimensional normed spaces

Definition 6.37

A finite dimensional normed space is a normed space $(V, \|\cdot\|)$ such that the vector space V is finite dimensional.

Example 6.38

$(\mathbb{K}^n, \|\cdot\|)$, where $\|\cdot\|$ is any norm on $\mathbb{K}^n$, is an n-dimensional normed space.

Recall (Analysis I, Tutorial 12.1) that a metric space (X, d) is called totally bounded if it has an ε-net for every $\varepsilon > 0$, that is if for any $\varepsilon > 0$ there exist a finite subset $\{a_1, \dots, a_N\}$ of X s.t.

$$X = \bigcup_{i=1}^{N} U_\varepsilon(a_i).$$

Proposition 6.39

Let us consider the normed space $(\mathbb{K}^n, \|\cdot\|_\infty)$ and $\emptyset \neq A \subseteq \mathbb{K}^n$. Then

(i) A is bounded $\Leftrightarrow$ A is totally bounded.

(ii) A is compact $\Leftrightarrow$ A is closed and bounded.

Proof

In the tutorial.

Corollary 6.40

For any $x \in \mathbb{K}^n$ and $r > 0$, the sphere $S_r(x) = \{y \in \mathbb{K}^n : \|y-x\|_\infty = r\}$ and the closed ball $B_r(x) = \{y \in \mathbb{K}^n : \|y-x\|_\infty \le r\}$ are compact in $(\mathbb{K}^n, \|\cdot\|_\infty)$.

Proof

Since obviously $S_r(x)$ and $B_r(x)$ are bounded, it remains to prove that they are closed in $(\mathbb{K}^n, \|\cdot\|_\infty)$. We have that $B_r(x)$ is closed, by Prop. 6.13 (iv).

Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $S_r(x)$ and $y = \lim_{n\to\infty} y_n$. Then $\lim_{n\to\infty} (y_n - x) = y - x$ and $\|y_n - x\| = 1$. By Lemma 6.12 (iii), we get that $\|y-x\| = \lim_{n\to\infty} \|y_n - x\| = 1$. Thus, $y \in S_r(x)$. Hence $S_r(x)$ is closed. $\square$

Proposition 6.41

Let $(V, \|\cdot\|)$ be an n-dimensional normed space and $b_1, \dots, b_n$ be a base of V. Then the function

$$T : \mathbb{K}^n \to V, \quad T(x) = \sum_{i=1}^n x_i b_i \quad \text{for every } x = (x_1, \dots, x_n) \in \mathbb{K}^n$$

is an isomorphism between the normed spaces $(\mathbb{K}^n, \|\cdot\|_\infty)$ and $(V, \|\cdot\|)$.

Proof

It is easy to see that T is a linear isomorphism. We shall prove that T is an isomorphism of normed spaces with the help of Proposition 6.34.

For every $x = (x_1, \dots, x_n) \in \mathbb{K}^n$ we obtain

$$\|T(x)\| = \left\| \sum_{i=1}^n x_i b_i \right\| \le \sum_{i=1}^n \|x_i b_i\| = \sum_{i=1}^n |x_i| \cdot \|b_i\| \le \sum_{i=1}^n \|x\|_\infty \cdot \|b_i\|$$
$$= \left(\sum_{i=1}^n \|b_i\| \right) \cdot \|x\|_\infty .$$

Hence, there exists $\beta := \sum_{i=1}^{n} \|b_i\| > 0$ s.t.

(1) $$\|T(x)\| \leq \beta \cdot \|x\|_\infty \quad \text{for every } x \in \mathbb{K}^n.$$

It follows that T is continuous and, furthermore, the function

$$f : \mathbb{K}^n \to \mathbb{R} \qquad f = T \circ \|\cdot\|$$

is also continuous (as a composition of continuous functions. Since the unit sphere $S_1(0) = \{x \in \mathbb{K}^n : \|x\|_\infty = 1\}$ is compact in $(\mathbb{K}^n, \|\cdot\|_\infty)$, by the Theorem of the Minimum and Maximum 3.52, we get an $a \in S_1(0)$ s.t. $f(x) \geq f(a)$ for all $x \in S_1(0)$. That is,

(2) $$\|T(x)\| \geq \|T(a)\| \quad \text{for all } x \in S_1(0).$$

Let $\alpha := \|T(a)\|$. Then $\alpha > 0$, since $\|a\|_\infty = 1$, so $a \neq 0$ and $T(a) \neq 0$.

On the other hand, for any $x \in \mathbb{K}^n$, $x \neq 0$ we get that $\left\| \frac{x}{\|x\|_\infty} \right\|_\infty = 1$, so $\frac{x}{\|x\|_\infty} \in S_1(0)$. By (2), $\left\| T\left(\frac{x}{\|x\|_\infty} \right) \right\| \geq \alpha$, which is equivalent with

(3) $$\|T(x)\| \geq \alpha \|x\|_\infty$$

Obviously, (3) is true also for $x = 0$.

From (1) and (3), it follows that T is an isomorphism of normed spaces. $\square$

<u>Theorem 6.42</u>

All n-dimensional normed spaces over $\mathbb{K}$ are isomorphic.

<u>Proof</u>

By Proposition 6.41 and Remark 6.34. $\square$

Theorem 6.43

All finite dimensional normed spaces are Banach spaces.

Proof

Let $(V, \|\cdot\|)$ be an n-dimensional normed space over $\mathbb{K}$. Then, by Proposition 6.41, $(V, \|\cdot\|)$ is isomorphic to $(\mathbb{K}^n, \|\cdot\|_\infty)$. By Theorem 6.16 (iv), $(\mathbb{K}^n, \|\cdot\|_\infty)$ is a Banach space. It is easy to see, using Proposition 6.34, that if we have two isomorphic normed spaces, one is a Banach space iff the other one is a Banach space. It follows that $(V, \|\cdot\|)$ is a Banach space. □

Theorem 6.44

All norms on a finite dimensional vector space are equivalent. In particular, all norms on $\mathbb{K}^n$ are equivalent.

Proof

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on an n-dimensional vector space V over $\mathbb{K}$. Let T be the function defined in Proposition 6.41 and T^{-1} be its inverse. Then $T : (V, \|\cdot\|_1) \to (\mathbb{K}^n, \|\cdot\|_\infty)$ is an isomorphism and $T^{-1} : (\mathbb{K}^n, \|\cdot\|_\infty) \to (V, \|\cdot\|_2)$ is also an isomorphism. It follows that $I_V = T^{-1} \circ T : (V, \|\cdot\|_1) \to (V, \|\cdot\|_2)$ is an isomorphism. By Corollary 6.36, we get that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. □

Theorem 6.45 (Bolzano–Weierstrass)

Let $(V, \|\cdot\|)$ be a finite dimensional normed space and $\phi \neq A \subseteq V$. Then

$$A \text{ is compact} \iff A \text{ is closed and bounded}.$$

Proof

By Proposition 6.38 and Theorem 6.42

$\square$

Proposition 6.46

Let V and W be normed spaces over $\mathbb{K}$. Assume that V is finite dimensional. Then any linear transformation $T: V \to W$ is continuous.

Proof

In the tutorial.

$\square$