6. Normed spaces

Definition 6.1

Let V be a vector space over $\mathbb{K}$, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. A norm on V is a function $\|\cdot\|: V \longrightarrow \mathbb{R}$ satisfying the following conditions:

(N1) $(\forall x \in V)\left(\|x\| \geqslant 0 \text{ and } (\|x\|=0 \Leftrightarrow x=0)\right)$

(N2) $(\forall \lambda \in \mathbb{K})(\forall x \in V)\left(\|\lambda \cdot x\| = |\lambda| \cdot \|x\|\right)$ (homogeneity)

(N3) $(\forall x, y \in V)\left(\|x+y\| \leq \|x\| + \|y\|\right)$ (triangle inequality).

A normed vector space over $\mathbb{K}$ is a structure $(V, \|\cdot\|)$, where V is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is a norm on V. Normed vector spaces are also called normed linear spaces or normed spaces. We shall often refer to V itself as a normed space when it is clear which norm is meant.

A normed space over $\mathbb{R}$ is also called a real normed space. Similarly, a normed space over $\mathbb{C}$ is called a complex normed space.

Remark 6.2

Every linear subspace of a normed space is a normed space with respect to the restriction of the norm.

Lemma 6.3

Let $(V, \|\cdot\|)$ be a normed space. Then the following are true:

(i) $(\forall x \in V)\left(\|-x\| = \|x\|\right)$.

(ii) $(\forall x \in V)\left(x \neq 0 \Rightarrow \left\|\frac{x}{\|x\|}\right\| = 1\right)$.

(iii) $(\forall n \in \mathbb{N})\,(\forall x_1, \dots, x_n \in V)\left(\left\|\sum_{i=1}^{n} x_i\right\| \leq \sum_{i=1}^{n} \|x_i\|\right)$

(iv) $(\forall x, y \in V)\left(\big|\, \|x\| - \|y\| \,\big| \leq \|x-y\|\right)$.

Proof

(i) $\|-x\| = \|(-1)\cdot x\| \overset{(N2)}{=\!=} |-1|\cdot\|x\| = 1\cdot\|x\| = \|x\|$.

(ii) Let $x \in V, x \neq 0$. Then by (N1) we have that $\|x\| \neq 0$, so

$\frac{1}{\|x\|} \in \mathbb{R}$. It follows that

$$\left\|\frac{x}{\|x\|}\right\| = \left\|\frac{1}{\|x\|}\cdot x\right\| \overset{(N2)}{=\!=} \left|\frac{1}{\|x\|}\right|\cdot\|x\| = \frac{1}{\|x\|}\cdot\|x\| = 1.$$

(iii) By induction on n.

$n=1$: Trivial.

$n \Rightarrow n+1$: Assume that $\left\|\sum_{i=1}^{n} x_i\right\| \leq \sum_{i=1}^{n} \|x_i\|$. We obtain

$$\left\|\sum_{i=1}^{n+1} x_i\right\| = \left\|\sum_{i=1}^{n} x_i + x_{n+1}\right\| \overset{(N3)}{\leq} \left\|\sum_{i=1}^{n} x_i\right\| + \|x_{n+1}\| \overset{(I.H.)}{\leq}$$
$$\leq \sum_{i=1}^{n} \|x_i\| + \|x_{n+1}\| = \sum_{i=1}^{n+1} \|x_i\|.$$

(iv) Let $x, y \in V$. Then

$$\big|\,\|x\| - \|y\|\,\big| \leq \|x-y\| \iff \|x\| - \|y\| \leq \|x-y\| \text{ and}$$
$$\|y\| - \|x\| \leq \|x-y\|.$$

We get that $\|x\| = \|(x-y)+y\| \overset{(N3)}{\leq} \|x-y\| + \|y\|$, so

$\|x\| - \|y\| \leq \|x-y\|$. By interchanging x and y, it follows that

$$\|y\|-\|x\| \leq \|y-x\| = \|-(x-y)\| \overset{(i)}{=\!=} \|x-y\|.$$

□

In the sequel, we give more examples of normed spaces.

Example 6.4

$(\mathbb{K}, |\cdot|)$ is a normed space, where $|\cdot|$ is the absolute value.

Example 6.5

For $x = (x_1, \dots, x_n) \in \mathbb{K}^n$ define

$$\|x\|_\infty := \max\{|x_1|, \dots, |x_n|\}$$

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} = \sqrt[p]{|x_1|^p + \dots + |x_n|^p} \quad \text{for } 1 \leq p < \infty.$$

Thus, $\|x\|_1 = |x_1| + \dots + |x_n|$, $\|x\|_2 = \sqrt{|x_1|^2 + \dots + |x_n|^2}$.

Then $(\mathbb{K}^n, \|\cdot\|_\infty)$ and $(\mathbb{K}^n, \|\cdot\|_p)$ $(1 \leq p < \infty)$ are normed spaces.

Proof.

Exercise. □

The norm $\|\cdot\|_2$ on $\mathbb{K}^n$ is called the Euclidean norm.

Example 6.6

Let $a < b \in \mathbb{R}$ and $\mathcal{C}([a,b])$ be the $\mathbb{R}$-vector space of all continuous functions $f : [a,b] \to \mathbb{R}$. For any $1 \leq p < \infty$ we define

$$\|\cdot\|_p : \mathcal{C}([a,b]) \to \mathbb{R}, \qquad \|f\|_p = \left(\int_a^b |f|^p\right)^{\frac{1}{p}}.$$

Then $(\mathcal{C}([a,b]), \|\cdot\|_p)$ $(1 \le p < \infty)$ is a normed space.

Proof

See Tutorial 6.

Example 6.7

(i) Let X be a set and let $\mathcal{B}(X)$ be the $\mathbb{K}$-vector space of all bounded functions $f: X \to \mathbb{K}$. Define

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}.$$

Then $(\mathcal{B}(X), \|\cdot\|_\infty)$ is a normed space.

(ii) Let (X,d) be a metric space and $\mathcal{C}(X) \cap \mathcal{B}(X)$ be the $\mathbb{K}$-vector space of all functions $f: X \to \mathbb{K}$ which are continuous and bounded. Then $(\mathcal{C}(X) \cap \mathcal{B}(X), \|\cdot\|_\infty)$ is a normed space.

(iii) Let (X,d) be a compact metric space. Then $(\mathcal{C}(X), \|\cdot\|_\infty)$ is a normed space.

Proof

(i) Similar with the proof of Remark 5.4.

(ii) It is easy to see that $\mathcal{C}(X) \cap \mathcal{B}(X)$ is a linear subspace of $\mathcal{B}(X)$. Hence, by Remark 6.2, $(\mathcal{C}(X) \cap \mathcal{B}(X), \|\cdot\|_\infty)$ is a normed space.

(iii) If (X,d) is compact, then by Theorem 3.52, $\mathcal{C}(X) \cap \mathcal{B}(X) = \mathcal{C}(X)$. Apply now (ii) to get that $(\mathcal{C}(X), \|\cdot\|_\infty)$ is a normed space. □

Example 6.8

Consider the set ℓ^∞ of all bounded sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{K}$ with

$$\|(x_n)\|_\infty := \sup\{|x_n| : n\in\mathbb{N}\}. \quad \text{for any } (x_n)\in\ell^\infty.$$

Then ℓ^∞ is a normed space.

Proof

$\ell^\infty = B(\mathbb{N})$. $\square$

Example 6.9

Let c be the set of all convergent sequences in $\mathbb{K}$ and c_0 be the set of sequences in $\mathbb{K}$ which converge to 0 with $\|\cdot\|_\infty$.

Then c, c_0 are normed spaces.

Proof

Remark that c, c_0 are linear subspaces of ℓ^∞ and apply Remark 6.2 $\square$

Proposition 6.10

Let $(V, \|\cdot\|)$ be a normed space and define

$$d : V\times V \to \mathbb{R}, \qquad d(x,y) = \|x-y\|.$$

Then d is a metric on V.

Proof

Let us verify that d satisfies the axioms of a metric. Let $x,y,z\in V$.

(i) $d(x,y) = \|x-y\| \overset{(N1)}{\geq} 0$ and $d(x,y)=0 \Leftrightarrow \|x-y\|=0 \overset{(N1)}{\Longleftrightarrow} x-y=0$

$\Leftrightarrow x=y$.

(ii) $d(y,x) = \|y-x\| \overset{L\,6.3(i)}{=\!=\!=} \|x-y\| = d(x,y)$.

(iii) $d(x,y) + d(y,z) = \|x-y\| + \|y-z\| \overset{(N3)}{\geq} \|(x-y)+(y-z)\| =$

$$= \|x-z\| = d(x,z).$$

□

The metric defined in Proposition 6.10 is referred to as the metric induced by the norm $\|\cdot\|$.

Thus, any normed space is a metric space with the metric induced by the norm.

Example 6.11

If we consider the normed space $(\mathbb{K}^n, \|\cdot\|_2)$, the induced metric is the Euclidean metric:

$$d(x,y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|} \quad \text{for } x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n).$$

Lemma 6.12

Let $(V, \|\cdot\|)$ be a normed space and d be the induced metric.

For any $x \in V$ and any sequence $(x_n)_{n\in\mathbb{N}}$ in V,

(i) $\|x\| = d(x,0)$.

(ii) $\lim_{n\to\infty} x_n = x \iff \lim_{n\to\infty} \|x_n - x\| = 0$ in $(\mathbb{R}, |\cdot|)$

(iii) $\lim_{n\to\infty} x_n = 0 \iff \lim_{n\to\infty} \|x_n\| = 0$.

(iv) $\lim_{n\to\infty} x_n = x \implies \lim_{n\to\infty} \|x_n\| = \|x\|$.

The converse is not true,

Proof

(i) $d(x,0) = \|x-0\| = \|x\|$.

(ii) $\lim_{n\to\infty} x_n = x$ in $(V,d) \iff \lim_{n\to\infty} d(x_n,x) = 0 \iff \lim_{n\to\infty} \|x_n - x\| = 0$.

(iii) Apply (ii) with $x=0$.

(iv) Assume that $\lim_{n\to\infty} x_n = x$. By Lemma 6.3 (iv), we have that

$$0 \le \big|\, \|x_n\| - \|x\| \,\big| \le \|x_n - x\|$$

Since $\lim_{n\to\infty} \|x_n - x\| = 0$, by (ii), it follows that $\lim_{n\to\infty} \|x_n\| = \|x\|$.

To see that the converse is not true, consider the normed space $(\mathbb{R}, |\cdot|)$ and the sequence $x_n = (-1)^n$. Then (x_n) is not convergent, but $\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} |(-1)^n| = \lim_{n\to\infty} 1 = 1$.

□

Proposition 6.13

Let $(V, \|\cdot\|)$ be a normed space. Then

(i) the mapping $\|\cdot\| : V \to \mathbb{K}$ is Lipschitz continuous.

(ii) if (x_n), (y_n) are sequences in V that converge to x and y respectively, and (α_n) is a sequence in $\mathbb{K}$ that converges to α, then (x_n+y_n) converges to $x+y$ and $(\alpha_n x_n)$ converges to αx.

(iii) if $\emptyset \ne A \subseteq V$, then A is bounded $\iff$ the set $\{\|x\| : x\in A\}$ is bounded in $\mathbb{R}$.

(iv) for all $x\in V$ and $r>0$, $\overline{U_r(x)} = B_r(x)$, where

$U_r(x) = \{y \in V : \|y-x\| < r\}$ and $B_r(x) = \{y\in V : \|y-x\| \le r\}$.

Proof

(i) By Lemma 6.3 (N) we get that for all $x, y \in V$,

$$\big| \|x\| - \|y\| \big| \le \|x-y\|,$$

hence $\|\cdot\|$ is Lipschitz continuous with Lipschitz constant 1.

(ii) For all $n \in \mathbb{N}$,

$$0 \le \|(x_n+y_n)-(x+y)\| = \|(x_n-x)+(y_n-y)\| \overset{(N3)}{\le} \|x_n-x\|+\|y_n-y\|.$$

Since $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, we get that $\lim_{n\to\infty}\|x_n-x\| =$ $= \lim_{n\to\infty}\|y_n-y\| = 0$. It follows that $\lim_{n\to\infty}\|(x_n+y_n)-(x+y)\|=0$, that is $\lim_{n\to\infty}(x_n+y_n) = x+y$.

Furthermore, for all $n \in \mathbb{N}$,

$$0 \le \|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\| =$$
$$= \|\alpha_n(x_n-x)+(\alpha_n-\alpha)x\| \overset{(N3)}{\le} \|\alpha_n(x_n-x)\| + \|(\alpha_n-\alpha)x\|$$
$$\overset{(N2)}{=} |\alpha_n|\cdot\|x_n-x\| + |\alpha_n-\alpha|\cdot\|x\|.$$

Since $\lim_{n\to\infty}\left(|\alpha_n|\cdot\|x_n-x\| + |\alpha_n-\alpha|\cdot\|x\|\right) = |\alpha|\cdot 0 + 0\cdot\|x\| = 0$, it follows that $\lim_{n\to\infty}(\alpha_n x_n) = \alpha x$.

(iii) Let $A \subseteq V$, $A \neq \emptyset$ and fix $a \in A$.

"$\Rightarrow$" If A is bounded, there exists $D>0$ s.t. for all $x, y \in A$

$$\|x-y\| = d(x,y) \le D.$$

It follows that for all $x \in V$,

$$\|x\| = \|x-a+a\| \le \|x-a\| + \|a\| \le D + \|a\|.$$

Take $M := D + \|a\|$. Then $\|x\| \le M$ for all $x \in A$, i.e. $\{\|x\| : x \in A\}$ is bounded.

$\Leftarrow$ Assume that there exists $M>0$ s.t. $\|x\| \le M$ for all $x \in A$.

Then for all $x,y \in A$,

$$d(x,y) = \|x-y\| = \|x+(-y)\| \le \|x\| + \|-y\| = \|x\| + \|y\| \le 2M.$$

Thus, A is bounded.

(iv)

„$\subseteq$" Let $y \in \overline{U_r(x)}$. Then there exists a sequence (y_n) in $U_r(x)$ s.t. $\lim_{n\to\infty} y_n = y$. We get that $\lim_{n\to\infty}(y_n - x) = y - x$ and, by Lemma 6.12(iv), $\lim_{n\to\infty} \|y_n - x\| = \|y-x\|$. Since $\|y_n - x\| < 1$ for all $n \in \mathbb{N}$, it follows that

$$\|y-x\| = \lim_{n\to\infty} \|y_n - x\| \le 1. \text{ Thus, } y \in B_r(x).$$

„$\supseteq$" Since $U_r(x) \subseteq \overline{U_r(x)}$, it remains to prove that $B_r(x) \setminus U_r(x) = \{y \in V : \|y-x\| = r\} \subseteq \overline{U_r(x)}$.

Let $y \in V$ s.t. $\|y-x\| = r$. Consider a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ s.t. $\lambda_n \in (0,1)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \lambda_n = 1$ (for example $\lambda_n = 1 - \frac{1}{n+1}$)

Define now $y_n := x + \lambda_n (y-x)$. Then

$$\|y_n - x\| = \|\lambda_n \cdot (y-x)\| = |\lambda_n| \cdot \|y-x\| = \lambda_n \cdot r < r,$$

so $y_n \in U_r(x)$ for all $n \in \mathbb{N}$.

$$\|y_n - y\| = \|x + \lambda_n(y-x) - y\| = \|(1-\lambda_n)x - (1-\lambda_n)y\| = \|(1-\lambda_n)(x-y)\|$$
$$= |1-\lambda_n| \cdot \|x-y\| = (1-\lambda_n) \cdot \|x-y\|.$$

It follows that $\lim_{n\to\infty} \|y_n - y\| = \lim_{n\to\infty} (1-\lambda_n)\cdot\|x-y\| = \|x-y\| \cdot \lim_{n\to\infty}(1-\lambda_n)$

$$= 0.$$

Thus, (y_n) is a sequence in $U_r(x)$ s.t. $\lim_{n\to\infty} y_n = y$. That is,

$y \in \overline{U_r(x)}$. $\square$

Definition 6.14

A **Banach space** is a normed space which is complete with the metric induced by the norm.

First examples of Banach spaces are $(\mathbb{R}, |\cdot|)$ and $(\mathbb{C}, |\cdot|)$.

Proposition 6.15

Let $X \subseteq Y$ be a subspace of a complete metric space Y. Then the following are equivalent:

(i) X is closed in Y

(ii) X is complete.

Proof

Exercise.

Theorem 6.16

(i) Let X be an arbitrary set. Then $(B(X), \|\cdot\|_\infty)$ is a Banach space.

(ii) If (X,d) is a metric space, then $(\mathcal{C}(X) \cap B(X), \|\cdot\|_\infty)$ is a Banach space

(iii) If (X,d) is a compact metric space, then $(\mathcal{C}(X), \|\cdot\|_\infty)$ is a Banach space.

(iv) $(\mathbb{K}^n, \|\cdot\|_\infty), n \geq 1$, is a Banach space

(v) ℓ^∞ is a Banach space.

Proof.

We have already proved that all these spaces are normed.

(i) Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $B(X)$. Since for all $x \in X$, we have that

$$|f_m(x) - f_n(x)| \leq \sup\{|f_m(y) - f_n(y)| : y \in X\} = \|f_n - f_m\|,$$

it follows that $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$.

From the fact that $\mathbb{K}$ is complete, we get that $(f_n(x))$ is convergent to a limit that depends on x, so we denote it with $f(x)$.

In this way, we define a function $f: X \to \mathbb{K}$. We shall prove that (f_n) converges towards f in $B(X)$, that is $f \in B(X)$ and $\lim_{n\to\infty} f_n = f$.

Let $\varepsilon > 0$. Since (f_n) is Cauchy, there exists $N \in \mathbb{N}$ s.t.

$$(\forall k, p > N)\left(\|f_k - f_p\|_\infty < \frac{\varepsilon}{2}\right).$$

Hence, $|f_k(x) - f_p(x)| < \frac{\varepsilon}{2}$ for all $x \in X$ and all $k, p > N$.

We get that

$$|f(x) - f_p(x)| = \lim_{k\to\infty} |f_k(x) - f_p(x)| \leq \frac{\varepsilon}{2} \quad \text{for all } x \in X, p > N.$$

Consequently,

$$(\forall p > N)\left(\|f - f_p\|_\infty = \sup\{f(x) - f_p(x) : x \in X\} \leq \frac{\varepsilon}{2} < \varepsilon\right).$$

Thus, $\lim_{p\to\infty} \|f - f_p\|_\infty = 0$, that is $\lim_{p\to\infty} f_p = f$.

It remains to prove that $f \in B(X)$, that is f is bounded.

Let $p > N$. Since f_p is bounded, there exists $M > 0$ s.t. $\|f_p\|_\infty \leq M$.

Then for all $x \in X$,

$$|f(x)| = |f(x) - f_p(x) + f_p(x)| \leq |f(x) - f_p(x)| + |f_p(x)| \leq$$

$$\leq \frac{\varepsilon}{2} + \|f_p\|_\infty \leq M + \frac{\varepsilon}{2}.$$

Thus, f is bounded.

(ii) Since $\mathcal{C}(X) \cap B(X) \subseteq B(X)$ and $(B(X), \|\cdot\|_\infty)$ is Banach, hence complete as a metric space, by Proposition 6.15 we have to prove that $\mathcal{C}(X) \cap B(X)$ is closed in $B(X)$.

Let $(f_n)_n$ be a sequence in $\mathcal{C}(X) \cap B(X)$ and $f = \lim_{n\to\infty} f_n$. We

We have to show that $f \in \mathcal{C}(X) \cap B(X)$, which is equivalent with $f \in \mathcal{C}(X)$, since the fact that $f \in B(X)$ follows from (i).

Let $x \in X$ and $\varepsilon > 0$. We have to find a $\delta > 0$ s.t.

(1) $$(\forall y \in X)\ \big(d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \big).$$

Since $\lim\limits_{n\to\infty} f_n = f$, there is $N \in \mathbb{N}$ s.t.

(2) $$(\forall n \geq N)\ \big(\| f_n - f \|_\infty < \tfrac{\varepsilon}{3} \big).$$

Let $n \geq N$. Then $f_n : X \to \mathbb{K}$ is continuous, so there exists $\delta > 0$ s.t.

(3) $$(\forall y \in X)\ \big(d(x,y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \tfrac{\varepsilon}{3} \big).$$

We prove now that (1) is satisfied with the above δ. Let $y \in X$ be s.t. $d(x,y) < \delta$. Then

$$\begin{aligned} |f(x) - f(y)| &= |(f(x) - f_n(x)) + (f_n(x) - f_n(y)) + (f_n(y) - f(y))| \leq \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\overset{(3)}{\leq} \|f - f_n\| + \frac{\varepsilon}{3} + \|f - f_n\| \overset{(2)}{<} 3 \cdot \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

(iii) follows immediately from (ii), since if (X,d) is a compact metric space, $\mathcal{C}(X) \cap B(X) = \mathcal{C}(X)$.

(iv) Let $X = \{1, \ldots, n\}$. Since any finite set is bounded, it follows immediately that each function $f : X \to \mathbb{K}$ is bounded, hence $\mathbb{K}^n = B(X)$.

Hence $(\mathbb{K}^n, \|\cdot\|_\infty) = (B(X), \|\cdot\|_\infty)$ is a Banach space, by (i).

(v) ℓ^∞ is defined in Example 6.8 and $\ell^\infty = B(\mathbb{N})$. Thus, ℓ^∞ is a Banach space.

$\square$

Example 6.17 (A normed space which is not a Banach space)

Consider the subset V of ℓ^∞ consisting of all sequences which have only finitely many non-zero terms.

$$V = \{ (x_n)_{n\in\mathbb{N}} \in \ell^\infty : (\exists N)(\forall n>N)(x_n = 0) \}.$$

It is easy to see that V is a linear subspace of ℓ^∞, hence by Remark 6.2, it is a normed space with $\|\cdot\|_\infty$.

For every $n\in\mathbb{N}$, let

$$x_n = \left(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots, \tfrac{1}{n}, 0, 0, 0, \dots\right) \in V.$$

Furthermore, let

$$x = \left(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots, \tfrac{1}{n}, \dots\right).$$

Then $\|x\|_\infty = \sup\{ \frac{1}{n} : n\in\mathbb{N}\} = 1$, hence $x\in\ell^\infty$. We have also that

$$\|x_n - x\|_\infty = \|(\underbrace{0, \dots, 0}_{n \text{ times}}, \tfrac{1}{n+1}, \dots, \tfrac{1}{n+1}, \dots)\| = \tfrac{1}{n+1} \to 0 \text{ as } n\to\infty.$$

Thus, $\lim_{n\to\infty} x_n = x$ in ℓ^∞, but $x\notin V$, while $x_n\in V$ for all $n\in\mathbb{N}$.

Hence, V is not closed in ℓ^∞ and, by Proposition 6.15, $(V, \|\cdot\|_\infty)$ is not a Banach space. $\square$

Equivalent norms

Definition 6.18

Let V be a $\mathbb{K}$-vector space. We say that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are <u>equivalent</u> if there are positive numbers $c, C \in \mathbb{R}_+$ s.t.

$$(\forall x \in V)\ \big(c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1 \big).$$

Remark 6.19

It is easy to check that equivalence of norms is an equivalence relation on the set of all norms on V.

Proof

Exercise. $\square$

Proposition 6.20

Let V be a $\mathbb{K}$-vector space. Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on V. For any subset $A \subseteq V$, $x \in V$ and any sequence (x_n) in V the following hold:

(i) (x_n) is Cauchy in $(V, \|\cdot\|_1) \Leftrightarrow (x_n)$ is Cauchy in $(V, \|\cdot\|_2)$

(ii) $\lim\limits_{n\to\infty} x_n = x$ in $(V, \|\cdot\|_1) \Leftrightarrow \lim\limits_{n\to\infty} x_n = x$ in $(V, \|\cdot\|_2)$

(iii) $(V, \|\cdot\|_1)$ is a Banach space $\Leftrightarrow (V, \|\cdot\|_2)$ is a Banach space

(iv) A is open (closed, bounded, compact, connected) in $(V, \|\cdot\|_1) \Leftrightarrow$ A is open (closed, bounded, compact, connected) in $(V, \|\cdot\|_2)$.

Proof.

Exercise. $\square$

Proposition 6.21

For all $x \in \mathbb{K}^n$,

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\,\|x\|_\infty.$$

Hence, $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are equivalent norms on $\mathbb{K}^n$.

Proof

Let $x = (x_1, \ldots, x_n) \in \mathbb{K}^n$. Since $\|x\|_\infty \geq |x_i|$ for all $i = 1, \ldots, n$, we get that

$$\|x\|_2^2 = \sum_{i=1}^n |x_i|^2 \leq \sum_{i=1}^n \|x\|_\infty^2 = n \cdot \|x\|_\infty^2, \text{ so } \|x\|_2 \leq \sqrt{n} \cdot \|x\|_\infty.$$

On the other hand,

$$\|x\|_\infty^2 = \max\{|x_i|^2 : i = 1, \ldots, n\} \leq \sum_{i=1}^n |x_i|^2 = \|x\|_2^2, \text{ so } \|x\|_\infty \leq \|x\|_2.$$

$\square$

Corollary 6.22

$(\mathbb{K}^n, \|\cdot\|_2)$ is a Banach space.

Proof

Since $(\mathbb{K}^n, \|\cdot\|_\infty)$ is a Banach space by Theorem 6.16 (iv), it follows from Proposition 6.21 and Proposition 6.20 (iii) that $(\mathbb{K}^n, \|\cdot\|_2)$ is also a Banach space.

$\square$