3. Rules of Integration

The Fundamental Theorem establishes a connection between integration and differentiation. This connection suggests techniques for integration which run parallel to prominent techniques for differentiation such as the chain rule and the product rule.

The following results elucidate this connection and play a significant role.

In the sequel, we use

Notation

If $b<a$ in $\mathbb{R}$ and $\varphi: [b,a] \to \mathbb{R}$ is integrable, then

$$\int_a^b \varphi \overset{\text{def}}{=\!=} -\int_b^a \varphi.$$

Substitution Theorem (compare with Theorem 5.26)

Let $f: I \to \mathbb{R}$ be a continuous function on the interval I and $u: [a,b] \to \mathbb{R}$ be a continuously differentiable function with $u([a,b]) \subseteq I$.

Then

$$\int_a^b f(u(t))\, u'(t)\, dt = \int_{u(a)}^{u(b)} f(x)\, dx.$$

Proof.

Since u is a continuous function defined on a compact interval $[a,b]$, the range of u is also a compact interval $[c,d]$. Thus, $u([a,b]) = [c,d] \subseteq I$.

Therefore we shall consider the restriction $f|_{[c,x]}$ of f to this interval.

Since $f: [c,d] \to \mathbb{R}$ is continuous, by the Fundamental theorem, f has an antiderivative $F: [c,d] \to \mathbb{R}$. Define

$$\phi : [a,b] \to \mathbb{R}, \quad \phi(t) = (F \circ u)(t) = F(u(t)).$$

F, u are differentiable, so the Chain Rule yields that ϕ is differentiable and for any $t \in [a,b]$,

$$\phi'(t) = F'(u(t)) \cdot u'(t) = f(u(t)) \cdot u'(t).$$

Since f, u, u' are continuous, the function ϕ' is also continuous. From the fact that ϕ is an antiderivative of ϕ' and the Fundamental Theorem applied to ϕ', it follows that ϕ' is integrable and

$$\phi(b) - \phi(a) = \int_a^b \phi'(t)\,dt = \int_a^b f(u(t)) \cdot f'(t)\,dt.$$

On the other hand,

$$\phi(b) - \phi(a) = F(u(b)) - F(u(a)) = \int_{u(a)}^{u(b)} f(x)\,dx,$$ where the last equality is obtained by applying the Fundamental theorem to f. $\square$

<u>Integration by Parts</u> (compare with Theorem 5.28)

Let $f: [a,b] \to \mathbb{R}$ be continuous, F be an antiderivative of f and $g: [a,b] \to \mathbb{R}$ be continuously differentiable. Then

$$\int_a^b f(x) g(x)\,dx = F(x)g(x)\Big|_a^b - \int_a^b F(x) g'(x)\,dx.$$

Proof.

Since F, g are differentiable, we get that $F \cdot g$ is differentiable and the Product Rule yields

$$(F \cdot g)' = F' \cdot g + F \cdot g' = f \cdot g + F \cdot g'.$$

Thus, $F \cdot g$ is an antiderivative of the function $\phi : [a,b] \to \mathbb{R}$,

$$\phi(x) = (f \cdot g + F \cdot g')(x) = f(x) g(x) + F(x) \cdot g'(x).$$

The function ϕ is continuous, so we can apply the Fundamental Theorem to get that ϕ is integrable and moreover

$$\int_a^b f(x) g(x)\, dx + \int_a^b F(x) \cdot g'(x)\, dx = \int_a^b \phi(x)\, dx = F(x) \cdot g(x) \Big|_a^b.$$

Thus,

$$\int_a^b f(x) g(x)\, dx = F(x) \cdot g(x) \Big|_a^b - \int_a^b F(x) \cdot g'(x)\, dx.$$

$\square$

Corollary

Let $f, g : [a,b] \to \mathbb{R}$ be continuously differentiable. Then

$$\int_a^b f'(x)\, g(x)\, dx = f(x) g(x) \Big|_a^b - \int_a^b f(x) \cdot g'(x)\, dx.$$

Example 1

$$\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \frac{\pi}{4}$$

We apply Integration by Parts with $f, g, F : [0, \frac{\pi}{2}] \to \mathbb{R}$, $f(x) = g(x) = \sin x$ and $F(x) = -\cos x$. We obtain

$$\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \int_0^{\frac{\pi}{2}} f(x)g(x)\,dx = F(x)g(x)\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} F(x)\cdot g'(x)\,dx =$$

$$= -\cos x \sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x)\cdot \cos x\,dx =$$

$$= -\cos x \sin x\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos^2 x\,dx = -\cos x \sin x\Big|_0^{\frac{\pi}{2}} +$$

$$+ \int_0^{\frac{\pi}{2}} (1-\sin^2 x)\,dx = \left(-\cos\frac{\pi}{2}\sin\frac{\pi}{2} + \cos 0 \sin 0\right) + \int_0^{\frac{\pi}{2}} 1\,dx -$$

$$- \int_0^{\frac{\pi}{2}} \sin^2 x\,dx = 0 + \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin^2 x\,dx.$$

Thus, $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin^2 x\,dx$, so $2\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{\pi}{2}$, that is

$$\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{\pi}{4}.$$

Example 2

$$\int_0^1 \frac{t}{1+t^2}\,dt = \frac{1}{2}\log 2.$$

We apply the Substitution Theorem. Define $u: [0,1] \to [1,2]$, $u(t) = 1+t^2$, $f: [1,2] \to \mathbb{R}$, $f(x) = \frac{1}{2x}$. Then $u'(t) = 2t$ and $\frac{t}{1+t^2} = \frac{1}{2u(t)}\cdot u'(t) =$ $= f(u(t))\cdot u'(t)$. Hence,

$$\int_0^1 \frac{t}{1+t^2}\,dt = \int_0^1 f(u(t))\cdot u'(t)\,dt = \int_{u(0)}^{u(1)} f(x)\,dx = \int_1^2 \frac{1}{2x}\,dx =$$

$$= \frac{1}{2}\int_1^2 \frac{1}{x}\,dx = \frac{1}{2}\log|x|\Big|_1^2 = \frac{1}{2}\log 2 - \frac{1}{2}\log 1 = \frac{1}{2}\log 2.$$

Example 3

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}.$$

We apply again the Substitution Theorem. Let $f: [0,1] \to \mathbb{R}$, $f(x) = \sqrt{1-x^2}$. Define $u: [0, \frac{\pi}{2}] \to [0,1]$, $u(t) = \sin t$ (we say also that we make the substitution $x = \sin t$). Then

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_{u(0)}^{u(\frac{\pi}{2})} f(x)\,dx = \int_0^{\frac{\pi}{2}} f(u(t))\cdot u'(t)\,dt = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\cdot \cos t\,dt$$

$$= \int_0^{\frac{\pi}{2}} |\cos t|\cdot \cos t\,dt = \int_0^{\frac{\pi}{2}} \cos^2 t\,dt = \int_0^{\frac{\pi}{2}} (1-\sin^2 t)\,dt =$$

$$= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin^2 t\,dt = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Example 4

Let $a, b > 0$, $a > b$. Compute $\int_a^b \log x\,dx$.

We apply Integration by Parts with $f, g, F : [a,b] \to \mathbb{R}$, $f(x) = 1$, $g(x) = \log x$, $F(x) = x$. We obtain

$$\int_a^b \log x\,dx = \int_a^b 1\cdot \log x\,dx = \int_a^b f(x) g(x) = F(x)g(x)\Big|_a^b - \int_a^b F(x)\cdot g'(x)\,dx$$

$$= x\log x\Big|_a^b - \int_a^b x\cdot\frac{1}{x}\,dx = x\log x\Big|_a^b - \int_a^b 1\,dx =$$

$$= x\log x\Big|_a^b - x\Big|_a^b = x(\log x - 1)\Big|_a^b = b(\log b - 1) - a(\log a - 1).$$

4. Improper Integrals

Recall that the definition of the Riemann integral $\int_a^b f(x)dx$ requires the function f to be bounded on a compact interval $[a,b]$.

In this section we will weaken these assumptions and get in this way the so-called improper integrals.

We consider three cases

I One limit of integration is infinite

Definition

Let $f: [a,\infty[\to \mathbb{R}$ be a function such that f is Riemann integrable on $[a,R]$ for every $R>a$. If the limit

$$\lim_{R\to\infty} \int_a^R f(x)\,dx$$

exists, we say that f is improperly integrable on $[a,\infty[$ or that the integral $\int_a^\infty f(x)dx$ converges and we write

$$\int_a^\infty f(x)\,dx := \lim_{R\to\infty} \int_a^R f(x)\,dx.$$

One defines analogously $\int_{-\infty}^a f(x)dx$ for a function $f:]-\infty,a] \to \mathbb{R}$.

Example 1

$$\int_1^\infty \frac{dx}{x^c} \text{ converges} \iff c>1.$$

In the case of convergence, $\int_1^\infty \frac{dx}{x^c} = \frac{1}{c-1}$.

Proof

Let $R>1$. For $c=1$ we get that $\int_1^R \frac{dx}{x} = \log x\Big|_1^R = \log R$ and, so,

$\lim_{R\to\infty} \int_1^R \frac{dx}{x} = \lim_{R\to\infty} \log R = \infty$. Thus, $\int_1^\infty \frac{dx}{x}$ diverges.

For $c \neq 1$, we obtain

$$\int_1^R \frac{dx}{x^c} = \int_1^R x^{-c}\,dx = \frac{x^{-c+1}}{-c+1}\Big|_1^R = \frac{1}{1-c}\left(R^{1-c}-1\right).$$

For $c>1$, we have that $c-1>0$, so $\lim_{R\to\infty} R^{1-c} = \lim_{R\to\infty}\left(\frac{1}{R}\right)^{c-1} = 0$.

Hence, $\lim_{R\to\infty}\int_1^R \frac{dx}{x^c} = \frac{1}{1-c}(0-1) = \frac{1}{c-1}$. It follows that for $c>1$

$\int_1^\infty \frac{dx}{x^c}$ converges and, moreover, $\int_1^\infty \frac{dx}{x^c} = \frac{1}{c-1}$.

For $c<1$, we have that $1-c>0$, so $\lim_{R\to\infty} R^{1-c} = \infty$. Thus,

$\lim_{R\to\infty}\int_1^R \frac{dx}{x^c} = \frac{1}{1-c}(\infty-1) = \infty$ and the integral $\int_1^\infty \frac{dx}{x^c}$ diverges.

□

II The function is not defined at a limit of integration

Definition

Let $f:]a,b] \to \mathbb{R}$ be a function such that f is integrable on $[a+\varepsilon, b]$ for every $0<\varepsilon<b-a$. If the limit

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_{a+\varepsilon}^b f(x)\,dx$$

exists, we say that f is improperly integrable on $]a,b]$ or that the integral $\int_a^b f(x)\,dx$ converges and we write

$$\int_a^b f(x)\,dx := \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_{a+\varepsilon}^b f(x)\,dx.$$

One defines analogously $\int_a^b f(x)\,dx$ for a function $f: [a,b[\to \mathbb{R}$.

Example 2

$\int_0^1 \frac{dx}{x^c}$ converges $\iff c<1$.

In the case of convergence, $\int_0^1 \frac{dx}{x^c} = \frac{1}{1-c}$.

Proof.

Let $0<\varepsilon<1$. For $c=1$, we have that $\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_\varepsilon^1 \frac{dx}{x} =$

$$= \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} -\log \varepsilon = -\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \log \varepsilon = -(-\infty) = +\infty.$$

Let now $c \neq 1$. Then $\int_\varepsilon^1 \frac{dx}{x^c} = \frac{x^{-c+1}}{-c+1}\Big|_\varepsilon^1 = \frac{1}{1-c}\left(1-\varepsilon^{-c+1}\right)$.

If $c>1$, $\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \varepsilon^{-c+1} = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \left(\frac{1}{\varepsilon}\right)^{c-1} = \infty$, so $\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_\varepsilon^1 \frac{dx}{x^c} = \infty$.

If $c<1$, $\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \varepsilon^{-c+1} = 0$, so $\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_\varepsilon^1 \frac{dx}{x^c} = \frac{1}{1-c}(1-0) = \frac{1}{1-c}$.

□

III Both limits of integration are critical

Definition

Let $f:]a,b[\to \mathbb{R}$, $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ be a function such that f is integrable on $[\alpha,\beta]$ for all α,β with $a<\alpha<\beta<b$.

Let $c \in]a,b[$. If both improper integrals

$$\int_a^c f(x)\,dx = \lim_{\substack{\alpha \to a \\ \alpha > a}} \int_\alpha^c f(x)\,dx \quad \text{and} \quad \int_c^b f(x)\,dx = \lim_{\substack{\beta \to b \\ \beta < b}} \int_c^\beta f(x)\,dx$$

converge, we say that f is <u>improperly integrable on $]a,b[$</u> or that $\int_a^b f(x)\,dx$ <u>converges</u> and we write

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Remark

The above definition does not depend on the point $c \in]a,b[$.

Proof

Exercise.

Example 3

$\int_0^\infty \frac{dx}{x^c}$ diverges for any $c \in \mathbb{R}$.

Proof

By Examples 1 and 2. □

The Integral Criterion

One of the most important applications of improper integrals arises in the context of infinite series. The idea is to compare improper integrals of the form $\int_1^\infty f(x)\,dx$ with infinite series $\sum_{n=1}^{\infty} a_n$ with nonnegative summands.

Lemma 1

Let $\varphi : [a, \infty[\to [0, \infty[$ be an increasing function. Then

(i) $\lim_{x\to\infty} \varphi(x) = \begin{cases} \|\varphi\| & \text{if } \varphi \text{ is bounded} \\ \infty & \text{otherwise} \end{cases}$

(ii) φ is bounded $\iff$ $(\varphi(n))_{n\geq a}$ is bounded

Proof.

See the tutorial.

Theorem (**The Integral Criterion**)

Let $f: [1,\infty[\longrightarrow [0,\infty[$ be a decreasing function. Then the infinite series $\sum_{n=1}^{\infty} f(n)$ converges $\iff$ $\int_1^{\infty} f(x)\,dx$ converges.

Proof

Since f is decreasing, we have that

$$f(k) \le f(x) \le f(k-1) \qquad \text{for } k \geq 2,\ k-1 \le x \le k.$$

Thus, for $k \geq 2$ we get

$$(1) \quad f(k) = \int_{k-1}^{k} f(k)\,dx \le \int_{k-1}^{k} f(x)\,dx \le \int_{k-1}^{k} f(k-1) = f(k-1).$$

By summing (1) for $k = 2, \dots, n$, where $n \geq 2$, it follows that

$$(2) \quad \sum_{k=2}^{n} f(k) \le \sum_{k=2}^{n} \int_{k-1}^{k} f(x)\,dx = \int_1^n f(x)\,dx \le \sum_{k=2}^{n} f(k-1) = $$
$$= \sum_{k=1}^{n-1} f(k).$$

Let us denote with (s_n) the sequence of partial sums of the series $\sum_{n=1}^{\infty} f(n)$, that is $s_n = \sum_{k=1}^{n} f(k)$.

Since f is decreasing and $f \geq 0$, it follows that f is integrable on $[1,R]$ for every $R>1$ and, moreover, $\int_1^R f(x)\,dx \geq 0$.

Hence, the function

$$\varphi: [1,\infty[\to [0,\infty[, \quad \varphi(R) = \int_1^R f(x)\,dx$$

is well-defined. It is easy to see that φ is increasing:

$$R_2 > R_1 \Rightarrow \varphi(R_2) = \int_1^{R_2} f(x)\,dx = \int_1^{R_1} f(x)\,dx + \int_{R_1}^{R_2} f(x)\,dx \geq \int_1^{R_1} f(x)\,dx =$$
$$= \varphi(R_1).$$

With these notations, (2) is equivalent with

(3) $$s_n - f(1) \le \varphi(n) \le s_{n-1} \quad \text{for every } n \geq 2.$$

„$\Rightarrow$" Assume that $\sum_{n=1}^{\infty} f(n)$ is convergent and let $s = \sum_{n=1}^{\infty} f(n)$.

We have that $s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n-1}$ and, since $f \geq 0$, (s_{n-1}) is increasing, so $s = \lim_{n\to\infty} s_{n-1} \geq s_{n-1}$ for all $n \geq 2$.

Then (3) yields $\varphi(n) \le s_{n-1} \le s$ for all $n \geq 2$, that is the sequence $(\varphi(n))_{n\geq 2}$ is bounded. We can apply now Lemma 1 to get that φ is bounded and, furthermore, that

$$\int_1^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_1^R f(x)\,dx = \lim_{R\to\infty} \varphi(R) = \|\varphi\|.$$

Thus, $\int_1^{\infty} f(x)\,dx$ converges.

„$\Leftarrow$" Assume now that $\int_1^{\infty} f(x)\,dx$ converges, that is $\lim_{R\to\infty} \varphi(R)$ exists. Applying again Lemma 1, it follows that φ is bounded and, hence, $(\varphi(n))$ is bounded.

By (3) we get that (s_n) is bounded. Since (s_n) is increasing, it follows that (s_n) is convergent. That is, $\sum_{n=1}^{\infty} f(n)$ converges.

<u>Example 4</u>

$$\sum_{n=1}^{\infty} \frac{1}{n^c} \text{ converges} \iff \int_1^{\infty} \frac{1}{x^c}\,dx \text{ converges} \iff c > 1.$$