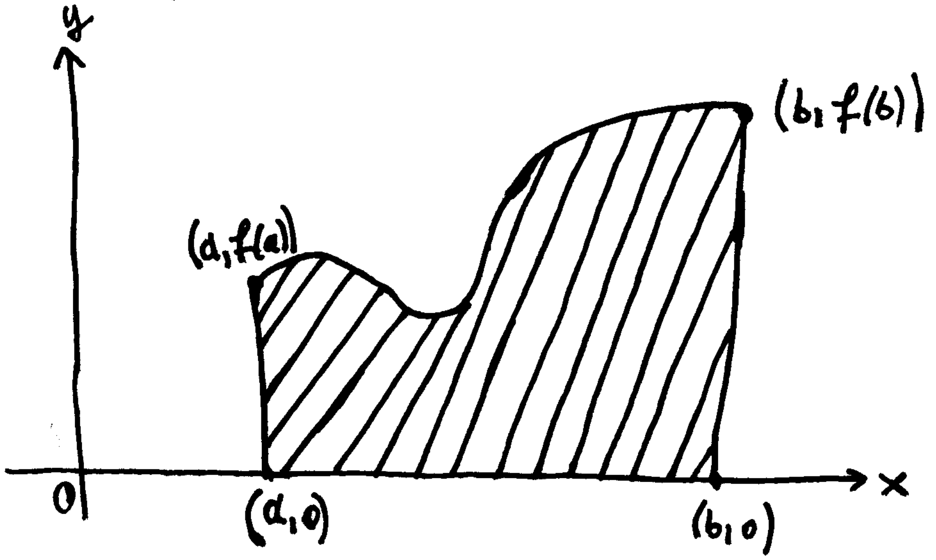
Chapter 5

Functions of one variable: Integrability

The problem which leads us to the concept of integration is totally different from the optimization problem which suggested differentiation.

Consider a function $f: [a,b] \to \mathbb{R}$. Our aim is to approximate the area bounded by the horizontal axis between $(a,0)$ and $(b,0)$, the vertical segments between $(a,0)$ and $(a, f(a))$ on the one hand and between $(b,0)$ and $(b, f(b))$ on the other, and finally by the graph G of the function f.



In general, we will encounter difficulties in measuring a given area under the graph of a function; this is at once illustrated by the graph of the characteristic function $f: [0,1] \to \mathbb{R}$ of the rational numbers:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

We first solve this problem for very special functions (so-called step functions).

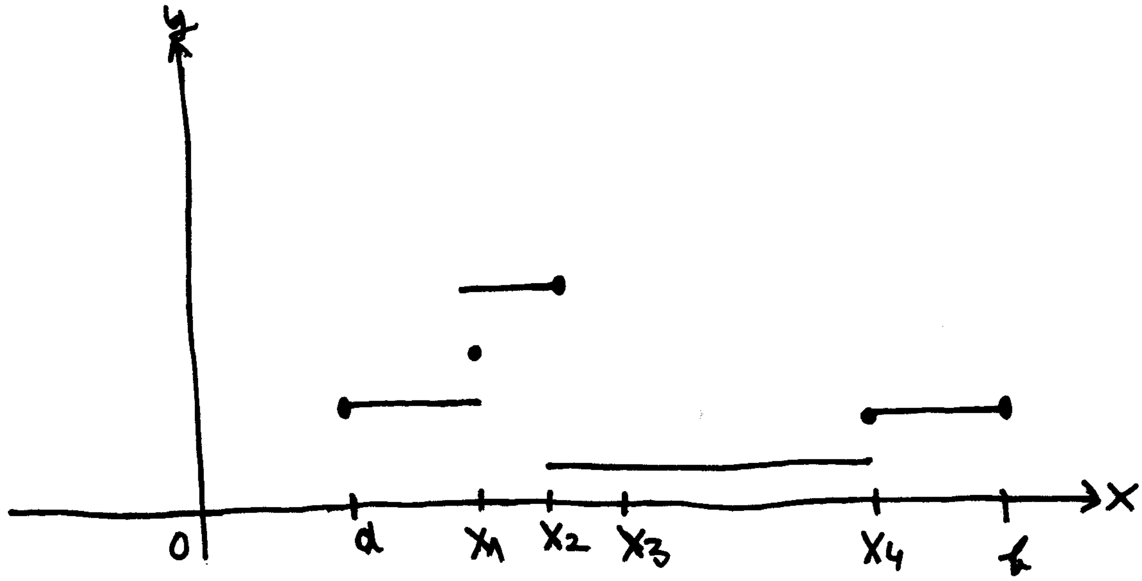
1. Definition of Integrability. Basics

We consider and fix a compact interval $[a,b]$, with $a<b$. In the set $\mathbb{R}^{[a,b]}$ of all functions $f: [a,b] \to \mathbb{R}$ we single out a subset which we shall call step functions.

Definition 5.1

A function $f: [a,b] \to \mathbb{R}$ is a step function iff there is a finite sequence $T = (a = x_0 < x_1 < \ldots < x_n = b)$ of real numbers such that f is constant on $]x_{k-1}, x_k[$ for all $k = 1, \ldots, n$. We say that the sequence T is a partition of $[a,b]$ associated with f.

The set of all step functions on $[a,b]$ will be denoted by $S = S[a,b] \subseteq \mathbb{R}^{[a,b]}$.



We should note that one and the same step function on $[a,b]$ has infinitely many associated partitions.

We say that a partition $T' = (a = y_0 < y_1 < \dots < y_p = b)$ <u>refines</u> a partition $T = (a = x_0 < x_1 < \dots < x_n = b)$ if $\{x_0, x_1, \dots, x_n\} \subseteq \{y_0, \dots, y_p\}$.

If T is a partition associated with a step function f, then any refinement T' of T is also associated with f.

If two partitions T_1 and T_2 of $[a,b]$ associated with f are given, there is always a partition which refines both of them. Just take the union of the points in T_1 and T_2 and relabel this finite set in ascending order to obtain the desired refinement.

<u>Remark 5.2</u>

(i) Every step function $f \in S[a,b]$ is bounded.

(ii) If f and g are step functions, then $f+g$, fg and, if defined, $\frac{f}{g}$ are step functions.

<u>Proof.</u>

(i) Let $f : [a,b] \to \mathbb{R}$ be a step function and $T = (a = x_0 < \dots < x_n = b)$ be a partition associated with f. For any $k = 1, \dots, n$ let us denote with c_k the value of f on $]x_{k-1}, x_k[$.

Then the image of f is the finite set

$$f([a,b]) = \{c_k : k = 1, \dots, n\} \cup \{f(a), f(x_1), \dots, f(x_{k-1}), f(b)\}.$$

Then $m := \min f([a,b])$ and $M := \max f([a,b])$ are the minimum resp. the maximum of f. Hence f is bounded.

(ii) Let T_f, T_g be two partitions of $[a,b]$ associated with f and g respectively. Let T be a common refinement of T_f and T_g. It is easy to see that T is a partition of $[a,b]$ associated with $f+g$, fg and - if defined - with $\frac{f}{g}$.

□

Define $\mathcal{B} = \mathcal{B}[a,b] := \{f : [a,b] \to \mathbb{R} : f \text{ is bounded}\}$. Then $S[a,b] \subseteq \mathcal{B}[a,b]$ and, by Theorem of the Minimum and Maximum, the set $\mathcal{C}[a,b] = \{f : [a,b] \to \mathbb{R} : f \text{ is continuous}\}$ is also contained in $\mathcal{B}[a,b]$.

Definition 5.3

Let $f, g : X \to \mathbb{R}$ be functions on a set X. We define the following functions

$$f \vee g,\ f \wedge g,\ f^+,\ f^-,\ |f| : X \to \mathbb{R}$$

by

(1) $(f \vee g)(x) \stackrel{\text{def}}{=} \max\{f(x), g(x)\}$,

(2) $(f \wedge g)(x) \stackrel{\text{def}}{=} \min\{f(x), g(x)\}$,

(3) $f^+(x) \stackrel{\text{def}}{=} \max\{f(x), 0\} = (f \vee 0)(x)$,

(4) $f^-(x) \stackrel{\text{def}}{=} \max\{-f(x), 0\} = ((-f) \vee 0)(x)$,

(5) $|f|(x) \stackrel{\text{def}}{=} |f(x)|$,

where $0 : X \to \mathbb{R}$ is the constant 0-function.

We call f^+ the positive part, f^- the negative part and $|f|$ the absolute value of f.

If f is bounded, we can associate with f a real number, called its <u>norm</u>, by

$$(6) \qquad \|f\| \overset{\text{def}}{=\!=} \sup\{\,|f(x)| : x \in X\}.$$

Note that the negative part of a function is always nonnegative. We should also keep in mind that the absolute value $|f|$ of a function is a function, while its norm $\|f\|$ is a number.

<u>Exercise E5.3</u>

Prove the following assertions:

(i) $|f| \geq 0$ and $|f| = 0$ exactly for $f = 0$.

(ii) $|f+g| \leq |f| + |g|$ and $|fg| = |f| \cdot |g|$.

(iii) $|f| = f^+ + f^-$ and $f = f^+ - f^-$.

<u>Remark 5.4</u>

For all $f, g \in B[a,b]$ we have

(i) $\|f\| \geq 0$ and $\|f\| = 0$ iff $f = 0$.

(ii) $\|f+g\| \leq \|f\| + \|g\|$ (Triangle Inequality)

(iii) $\|\lambda f\| = |\lambda| \cdot \|f\|$ for all $\lambda \in \mathbb{R}$.

(iv) $\|f \cdot g\| \leq \|f\| \cdot \|g\|$.

<u>Proof.</u>

Exercise $\square$

Since we have $S[a,b] \subseteq B[a,b]$, all the above statements hold for step functions.

Area below the graph of a step function

Now we consider a step function $f: [a,b] \to \mathbb{R}$ and "compute" the area "below" its graph G. In principle, this should be easy since we have to add the areas of rectangles only.

However, we have to recall that one function gives rise to many different partitions. We must therefore prove first that the counting of areas of rectangles produces a number which depends on the function only, but not on an associated partition of its domain.

Let I be a bounded non-empty interval with $s = \inf I$ and $t = \sup I$. We set $m(I) \overset{\text{def}}{=\!=} t - s$ and call this number the length or the measure of I. If $I = \emptyset$ then we set $m(I) = 0$.

Let M be a disjoint union of a finite set of intervals $I_1, \dots, I_k$. Then we define

$$m(M) \overset{\text{def}}{=\!=} \sum_{j=1}^{k} m(I_j).$$

However, this definition does require the verification of its independence of the representation of one and the same set as different disjoint unions of finitely many intervals.

This follows from the following Lemma.

Lemma 5.5

(i) If $I_1, \ldots, I_r$ and $J_1, \ldots, J_s$ are disjoint families of intervals and $M = \bigcup_{j=1}^{r} I_j = \bigcup_{k=1}^{s} J_k$, then

$$\sum_{j=1}^{r} m(I_j) = \sum_{k=1}^{s} m(J_k).$$

Therefore, $m(M)$ is uniquely associated with M and is independent of the representation of M as a disjoint union of finitely many intervals.

(ii) Let $f: [a,b] \to \mathbb{R}$ be a step function and $T_1 = (a = x_0 < \ldots < x_n = b)$, $T_2 = (a = y_0 < \ldots < y_p = b)$ be two partitions of $[a,b]$ associated with f. For $j = 1, \ldots, n$ and $k = 1, \ldots, p$ we select arbitrarily points $\xi_j \in]x_{j-1}, x_j[$ and $\eta_k \in]y_{k-1}, y_k[$. Then

$$\sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}) = \sum_{k=1}^{p} f(\eta_k)(y_k - y_{k-1})$$
$$= \sum_{w \in W} w \cdot m(f^{-1}(w)),$$

where $W = f([a,b])$ is the finite image of f.

Proof.

(i) Since $I_j \subseteq M$, $J_k \subseteq M$ for every $j = 1, \ldots, r$, $k = 1, \ldots, s$, we get that

(1) $I_j = I_j \cap M = I_j \cap \left(\bigcup_{k=1}^{s} J_k \right) = \bigcup_{k=1}^{s} (I_j \cap J_k)$,

(2) $J_k = J_k \cap M = J_k \cap \left(\bigcup_{j=1}^{r} I_j \right) = \bigcup_{j=1}^{r} (J_k \cap I_j)$.

Since I_j is an interval and $I_j \cap J_1, \ldots, I_j \cap J_s$ is a disjoint family of intervals, it is easy to see that (1) implies $m(I_j) = \sum_{k=1}^{s} m(I_j \cap J_k)$.

We get similarly that (2) implies $m(J_k) = \sum_{j=1}^{r} m(I_j \cap J_k)$.

It follows that

$$\sum_{j=1}^{r} m(I_j) = \sum_{j=1}^{r} \sum_{k=1}^{s} m(I_j \cap J_k) = \sum_{k=1}^{s} \sum_{j=1}^{r} m(I_j \cap J_k)$$
$$= \sum_{k=1}^{s} m(J_k).$$

(ii) It suffices to show that

$$\sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}) = \sum_{w \in W} w \cdot m(f^{-1}(w)),$$

since the right hand side is independent of partitions.

For a proof of this claim let $w \in W$. Let us denote with $J(w)$ the set of indices j with $f|_{]x_{j-1}, x_j[} = w$ and with $K(w)$ the set of indices j with $f(x_j) = w$. Then

$$f^{-1}(w) = \bigcup_{j \in J(w)}]x_{j-1}, x_j[\ \cup \bigcup_{j \in K(w)} \{x_j\}.$$

Thus, $f^{-1}(w)$ is a disjoint union of a finite family of intervals, so

$$m(f^{-1}(w)) = \sum_{j \in J(w)} m(]x_{j-1}, x_j[) + \sum_{j \in K(w)} m(\{x_j\}) = \sum_{j \in J(w)} (x_j - x_{j-1}),$$

because for any $j \in K(w)$ we have that $\{x_j\} = [x_j, x_j]$ is a one-element interval and $m(\{x_j\}) = 0$.

It follows that

$$w \cdot m(f^{-1}(w)) = w \cdot \sum_{j \in J(w)} (x_j - x_{j-1}) = \sum_{j \in J(w)} w \cdot (x_j - x_{j-1})$$
$$= \sum_{j \in J(w)} f(\xi_j)(x_j - x_{j-1}),$$

so

$$\sum_{w\in W} w\cdot m(f^{-1}(w)) = \sum_{w\in W}\sum_{j\in\partial(w)} f(\xi_j)(x_j-x_{j-1}) = \sum_{j=1}^{n} f(\xi_j)(x_j-x_{j-1}),$$

since $\{1,\dots,n\} = \bigcup_{w\in W} \partial(w)$ and for $w\neq w'$ in W, $\partial(w)\cap\partial(w')=\phi$. □

Definition 5.6

The number $\sum_{w\in W} w\cdot m(f^{-1}(w))$ from Lemma 5.5 (which is independent of any partition associated with f) is called the integral of f and is written $\int_a^b f(x)\,dx$ or - more short - $\int_a^b f$ or even $\int f$.

Remark

If $f:[a,b]\to\mathbb{R}$, $f(x)=c$ is a constant function, then

$\int_a^b f = c(b-a)$.

Proposition 5.7

Let $f,g:[a,b]\to\mathbb{R}$ be step functions and $r\in\mathbb{R}$. Then

(i) $\int(f+g)=\int f+\int g$, $\int(rf)=r\cdot\int f$ and $\int(f-g)=\int f-\int g$.

(ii) If $f\leq g$, then $\int f\leq\int g$.

(iii) $0\leq\left|\int f\right|\leq\int|f|\leq\|f\|(b-a)$.

Proof.

(i) Let $T=(a=x_0<\dots<x_n=b)$ be a partition of $[a,b]$ associated both with f and g. Let $\xi_j\in\,]x_{j-1},x_j[$, $j=1,\dots,n$ be

arbitrary points. Then, by Lemma 5.5,

$$S(f+g) = \sum_{j=1}^{n} (f+g)(\xi_j)(x_j - x_{j-1}) = \sum_{j=1}^{n} (f(\xi_j) + g(\xi_j))(x_j - x_{j-1}) =$$

$$= \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}) + \sum_{j=1}^{n} g(\xi_j)(x_j - x_{j-1}) =$$

$$= Sf + Sg.$$

We get similarly that

$$S(rf) = \sum_{j=1}^{n} (rf)(\xi_j)(x_j - x_{j-1}) = \sum_{j=1}^{n} r \cdot f(\xi_j)(x_j - x_{j-1}) =$$

$$= r \cdot \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}) = r \cdot Sf.$$

As a consequence,

$$S(f-g) = S(f + (-g)) = Sf + S(-g) = Sf - Sg.$$

(ii) Let $T = (a = x_0 < \ldots < x_n = b)$ and $\xi_j \in]x_{j-1}, x_j[$, $j = 1, \ldots, n$ be as in (i). Applying again Lemma 5.5 we get

$$Sf = \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}) \leq \sum_{j=1}^{n} g(\xi_j)(x_j - x_{j-1}) = Sg.$$

(iii) Let $T = (a = x_0 < \ldots < x_n = b)$ be a partition of $[a,b]$ associated with f and $\xi_j \in]x_{j-1}, x_j[$ for $j = 1, \ldots, n$. Then

$$0 \leq |Sf| = \left| \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}) \right| \leq \sum_{j=1}^{n} |f(\xi_j)|(x_j - x_{j-1}) =$$

$$= S|f|$$

Moreover,

$$S|f| = \sum_{j=1}^{n} |f(\xi_j)| \cdot (x_j - x_{j-1}) \leq \sum_{j=1}^{n} \|f\| \cdot (x_j - x_{j-1}) =$$

$$= \|f\| \cdot \sum_{j=1}^{n} (x_j - x_{j-1}) = \|f\| \cdot (b-a).$$

□

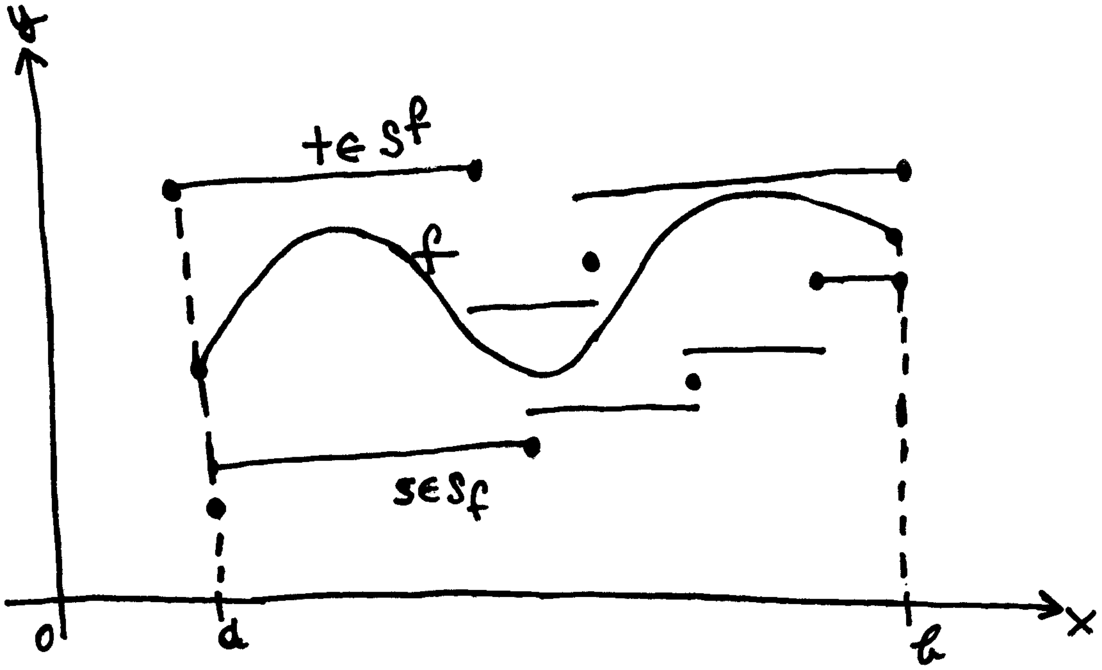
Integrability of functions in general

Now we return to an arbitrary function $f: [a,b] \to \mathbb{R}$ which we assume to be bounded.

Define

S_f = the set of step functions $s \in S[a,b]$ with $s \le f$,

S^f = the set of step functions $t \in S[a,b]$ with $f \le t$.



Proposition 1

Let $f: [a,b] \to \mathbb{R}$ be a bounded function. Then

(i) For any $s \in S_f$, $\int s \le \|f\|(b-a)$. Thus, $\sup\{\int s : s \in S_f\}$ exists.

(ii) For any $t \in S^f$, $-\|f\|(b-a) \le \int t$. Thus, $\inf\{\int t : t \in S^f\}$ exists.

Proof.

Since f is bounded, $\|f\| := \sup\{|f(x)| : x \in [a,b]\}$ exists and

$$-\|f\| \le f(x) \le \|f\| \quad \text{for all } x \in [a,b].$$

Let us consider the constant functions $c, C : [a,b] \to \mathbb{R}$,

$c(x) = -\|f\|$, $C(x) = \|f\|$ for all $x \in [a,b]$.

Then $c \le f \le C$, so $c \in S_f$ and $C \in S^f$.

(i) For any $s \in S_f$ we have that $s \le f \le C$. Hence, by Prop. 5.7 (ii), we get that $\int s \le \int C = \|f\|(b-a)$. Thus, $\{\int s : s \in S_f\}$ is a non-empty set of real numbers with an upper bound. Applying the Least Upper Bound Axiom 1.31, $\sup\{\int s : s \in S_f\}$ exists.

(ii) Similarly. $\square$

Definition of the Riemann integral

Definition 5.11

(i) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then $\sup\{\int s : s \in S_f\}$ is called the lower integral of f and denoted by $\underline{\int}_a^b f$ or $\underline{\int} f$.

Analogously, $\inf\{\int t : t \in S^f\}$ is called the upper integral of f and denoted by $\overline{\int}_a^b f$ or $\overline{\int} f$.

(ii) A function $f : [a,b] \to \mathbb{R}$ is called integrable in the sense of Riemann (or (Riemann) integrable) iff f is bounded and $\underline{\int} f = \overline{\int} f$. In this case, the number $\underline{\int} f = \overline{\int} f$ is called the Riemann integral of f and it is denoted by $\int_a^b f(x)\,dx$ or $\int_a^b f$ or $\int f$.

The space of all Riemann integrable functions $f : [a,b] \to \mathbb{R}$ will be denoted by $I = I[a,b]$.

Remark

For any bounded function $f:[a,b]\to\mathbb{R}$, $\underline{\int} f \le \overline{\int} f$.

Remark

Each step function is integrable and its Riemann integral coincides with the one defined in 5.6.

The condition of integrability can be reformulated so that an extremely applicable criterion emerges.

Theorem 5.12 (Riemann Criterion for integrability)

For a function $f:[a,b]\to\mathbb{R}$, the following statements are equivalent:

(i) f is integrable.

(ii) For every $\varepsilon>0$ there are step functions $s,t\in S[a,b]$ s.t. $s\le f\le t$ and $\int t-\int s=\int(t-s)<\varepsilon$.

Proof.

(i) $\Rightarrow$ (ii) Assume that f is integrable and let $\varepsilon>0$. By Definition 5.11 and the Characterization Theorem for suprema (and infima) 1.30 there are step functions $s,t\in S[a,b]$ s.t. $s\le f\le t$ and

$$\int f-\frac{\varepsilon}{2}<\int s\le\int f\le\int t<\int f+\frac{\varepsilon}{2}.$$

Hence, $\int(t-s)\overset{5.7(i)}{=\!=}\int t-\int s<\left(\int f+\frac{\varepsilon}{2}\right)-\left(\int f-\frac{\varepsilon}{2}\right)=\varepsilon.$

(ii) $\Rightarrow$ (i) Let $\varepsilon>0$ be arbitrary and s,t as in (ii). Then $s\le f\le t$

and the boundedness of step functions implies the boundedness of f. Thus $\underline{\int} f$ and $\overline{\int} f$ exist and one has

$$\int s \leq \underline{\int} f \leq \overline{\int} f \leq \int t.$$

Hence,

$$0 \leq \overline{\int} f - \underline{\int} f \leq \int t - \int s = \int (t-s) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, then $\underline{\int} f = \overline{\int} f$ and f is integrable.

$\square$

Example of a function which is not integrable

Let us consider the function

$$f : [0,1) \to \mathbb{R}, \quad f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\underline{\int} f = 0$, $\overline{\int} f = 1$, so f is not integrable.

As an application of the Riemann Criterion we get

Theorem 5.13

Every monotone function $f : [a,b] \to \mathbb{R}$ is integrable.

Proof.

W.l.o.g. we may assume that f is isotone. Since $f(a) \leq f(x) \leq f(b)$ for all $x \in [a,b]$, f is bounded.

Let $n \in \mathbb{N}$ and consider the partition $P_n = (a = x_0 < \dots < x_n = b)$, where

$$x_k = a + k \cdot \frac{b-a}{n}, \quad k = 0, 1, \dots, n.$$

We define step functions $s, t \in S[a,b]$ with which the partition is associated as follows:

$$s(x) \overset{\text{def}}{=\!=} f(x_{k-1}), \quad t(x) \overset{\text{def}}{=\!=} f(x_k) \quad \text{for } x \in [x_{k-1}, x_k[,$$

$$s(b) = t(b) = f(b).$$

Since f is isotone, we have that $s \leq f \leq t$. Moreover,

$$\int (t-s) = \int t - \int s = \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}) - \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} \big(f(x_k) - f(x_{k-1})\big)(x_k - x_{k-1}) = \sum_{k=1}^{n} \big(f(x_k) - f(x_{k-1})\big) \cdot \frac{(b-a)}{n}$$

$$= \frac{b-a}{n} \sum_{k=1}^{n} \big(f(x_k) - f(x_{k-1})\big) = \frac{b-a}{n}\big(f(b) - f(a)\big).$$

For every $\varepsilon > 0$, if we chose $n \in \mathbb{N}$ s.t. $\frac{b-a}{n} \cdot \big(f(b) - f(a)\big) < \varepsilon$, that is $n > \dfrac{(b-a)\big(f(b) - f(a)\big)}{\varepsilon}$, then $\int (t-s) < \varepsilon$.

Thus, condition (ii) of the Riemann Criterion is verified, so we can conclude that f is integrable.

The case of antitone functions is treated similarly. □

<u>Corollary</u>

(i) For $a \geq 0$,

$$p_m : [a,b] \to \mathbb{R}, \quad p_m(x) = x^m$$

is integrable for each $m \in \mathbb{N}_0$.

(ii) The restriction $\exp : [a,b] \to \mathbb{R}$ is integrable.

Proposition 5.14-15

Let $f, g : [a,b] \to \mathbb{R}$ be bounded functions and $r \in \mathbb{R}$. Then

(i) $\underline{\int} f = -\overline{\int}(-f)$ and $\overline{\int} f = -\underline{\int}(-f)$.

(ii) $\underline{\int}(f+g) \geq \underline{\int} f + \underline{\int} g$ and $\overline{\int}(f+g) \leq \overline{\int} f + \overline{\int} g$. If f, g are integrable, then $f+g$ is integrable and $\int(f+g) = \int f + \int g$.

(iii) If $r \geq 0$, then $\underline{\int}(rf) = r \cdot \underline{\int} f$ and $\overline{\int}(rf) = r \cdot \overline{\int} f$. If $r < 0$, then $\underline{\int}(rf) = r \cdot \overline{\int} f$ and $\overline{\int}(rf) = r \cdot \underline{\int} f$. If f is integrable, then rf is integrable and $\int(rf) = r \cdot \int f$.

(iv) $\underline{\int}(f \vee g) \geq \underline{\int} f \vee \underline{\int} g$ and $\overline{\int}(f \vee g) \geq \overline{\int} f \vee \overline{\int} g$. If f, g are integrable, then $f \vee g$ is integrable and $\int(f \vee g) \geq \int f \vee \int g$.

(v) If f, g are integrable and $f \leq g$, then $\int f \leq \int g$.

(vi) f is integrable if and only if f^+ and f^- are integrable. In that case, $\int f = \int f^+ - \int f^-$ and $\int |f| = \int f^+ + \int f^-$.

(vii) $|\underline{\int} f|, |\overline{\int} f| \leq \|f\|(b-a)$.

(viii) If f is integrable, then

$$|\int f| \leq \int |f| \leq \|f\|(b-a).$$

(ix) If f is integrable, then f^2 is integrable.

(x) If f, g are integrable, then $f \cdot g$ is integrable.

Proof.

(i) $-\overline{\int}(-f) = -\inf\{\int t : t \in S^{-f}\} = -\inf\{\int t : t$ is a step function and $t \geq -f\} = -\inf\{\int t : t$ is a step function and $-t \leq f\}$

$= -\inf\{-\int(-t) : t$ is a step function with $-t \le f\} =$

$= -\inf\{-\int s : s$ is a step function with $s \le f\} = -\inf\{-\int s : s \in S_f\}$

$= \sup\{\int s : s \in S_f\} = \underline{\int} f$, since for any subset $A \subseteq \mathbb{R}$, we have that $\sup A = -\inf(-A)$.

(ii) Since $\{s+t : s \in S_f \wedge t \in S_g\} \subseteq \{r : r \in S_{f+g}\}$, we get

$$\underline{\int}(f+g) = \sup\{\int r : r \in S_{f+g}\} \geq \sup\{\int(s+t) : s \in S_f \wedge t \in S_g\}$$

$$= \sup\{\int s : s \in S_f\} + \sup\{\int t : t \in S_g\} = \underline{\int} f + \underline{\int} g.$$

Similarly, $\{s+t : s \in S^f \wedge t \in S^g\} \subseteq \{r : r \in S^{f+g}\}$, so

$$\overline{\int}(f+g) = \inf\{\int r : r \in S^{f+g}\} \leq \inf\{\int(s+t) : s \in S^f \wedge t \in S^g\}$$

$$= \inf\{\int s : s \in S^f\} + \inf\{\int t : t \in S^g\} = \overline{\int} f + \overline{\int} g.$$

If f, g are integrable, then $\underline{\int} f = \overline{\int} f$ and $\underline{\int} g = \overline{\int} g$. It follows that

$$\int f + \int g = \underline{\int} f + \underline{\int} g \leq \underline{\int}(f+g) \leq \overline{\int}(f+g) \leq \overline{\int} f + \overline{\int} g = \int f + \int g.$$

Hence, $\underline{\int}(f+g) = \overline{\int}(f+g) = \int f + \int g$, so $f+g$ is integrable and $\int(f+g) = \int f + \int g$.

(iii) For $r \geq 0$, $\{rs : s \in S_f\} = \{t : t \in S_{rf}\}$, $\{rs : s \in S^f\} = \{t : t \in S^{rf}\}$.

Then $\underline{\int}(rf) = \sup\{\int t : t \in S_{rf}\} = \sup\{\int(rs) : s \in S_f\} =$

$$\overset{(5.7)}{=} \sup\{r \cdot \int s : s \in S_f\} = r \cdot \sup\{\int s : s \in S_f\} = r \cdot \underline{\int} f.$$

We prove similarly that $\overline{\int}(rf) = r \cdot \overline{\int} f$.

For $r < 0$, by (i) and the fact that $-r > 0$, we get that

$\underline{S}(rf) = -\overline{S}(-r)\cdot f = -(-r)\,\overline{S}f = r\cdot\overline{S}f$ and similarly

$\overline{S}(rf) = r\cdot\underline{S}f$.

The fact that f integrable implies rf integrable and $S(rf) = r\cdot Sf$ follows as in (ii).

(iv) Since $\{s\vee t : s\in S_f \wedge t\in S_g\} \subseteq \{r : r\in S_{f\vee g}\}$, we get similarly that

$$\underline{S}(f\vee g) = \sup\{Sr : r\in S_{f\vee g}\} \geq \sup\{Ss : s\in S_f\}\vee\sup\{St : t\in S_g\}$$
$$= \underline{S}f \vee \underline{S}g.$$

Using now that $\{r : r\in S^{f\vee g}\} \subseteq \{s : s\in S^f\}\cap\{t : t\in S^g\}$, it follows that

$$\overline{S}(f\vee g) = \inf\{Sr : r\in S^{f\vee g}\} \geq \inf\{Ss : s\in S^f\}\vee\inf\{St : t\in S^g\}$$
$$= \overline{S}f \vee \overline{S}g.$$

For the second part we use the Riemann Criterion and the following lemma on real numbers:

<u>The Four Lemma</u> Let $a,b,c,d\in\mathbb{R}$ and assume that $a\leq b$ and $c\leq d$.

Then $(b\vee d) - (a\vee c) \leq (b-a) + (d-c)$.

<u>Proof.</u> Exercise.

Assume that f,g are integrable. Applying the Riemann Criterion, we get step functions $s_f, t_f, s_g, t_g \in S[a,b]$ s.t. $s_f\leq f\leq t_f$, $s_g\leq g\leq t_g$ and $0\leq S(t_f - s_f),\ S(t_g - s_g) < \frac{\varepsilon}{2}$.

Define $s = s_f \vee s_g$ and $t = t_f \vee t_g$. Then s, t are also step functions, $s \le f \vee g \le t$ and $0 \le t - s \le (t_f - s_f) + (t_g - s_g)$ by the Four Lemma.

Applying Prop. 5.7, it follows that

$$0 \le \int (t-s) \le \int (t_f - s_f) + \int (t_g - s_g) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

A final application of the Riemann Criterium yields the integrability of $f \vee g$. The fact that $\int (f \vee g) \geq \int f \vee \int g$ follows immediately.

(v) Since $f \le g$, we have that $f \vee g = g$. By (iv), it follows that

$$\int g = \int (f \vee g) \ge \int f \vee \int g \geq \int f.$$

(vi)

„$\Rightarrow$" Assume that f is integrable. Since the constant 0-function is also integrable and $f^+ = f \vee 0$, $f^- = (-f) \vee 0$, (iii) and (iv) yield the integrability of f^+ and f^-.

„$\Leftarrow$" Assume that f^+ and f^- are integrable. By Ex. 5.3 (iii), $f = f^+ - f^-$, $|f| = f^+ + f^-$. Now, applying (ii) and (iii) we get that f and $|f|$ are integrable and, moreover, $\int f = \int f^+ - \int f^-$, $\int |f| = \int f^+ + \int f^-$.

(vii) follows immediately from (the proof of) Prop. 1 (before the definition of the Riemann integral).

(viii) Since $|f| \le \|f\|$, (v) yields

$$\int |f| \le \int \|f\| = \|f\| (b-a).$$

Moreover,

$$\left|\int f\right| = \left|\int (f^+ - f^-)\right| = \left|\int f^+ - \int f^-\right| \le \left|\int f^+\right| + \left|\int f^-\right| =$$

$= \int f^+ + \int f^- = \int |f|$, since $f^+, f^- \geq 0$ and so, by (v),

$$\int f^+, \int f^- \geq 0.$$

(ix) Assume first that $f \geq 0$. Let $\varepsilon > 0$ be given. Since f is integrable, we can apply the Riemann Criterion to find step functions $s_0, t_0 \in S[a,b]$ s.t. $s_0 \leq f \leq t_0$ and

$$\int (t_0 - s_0) < \frac{\varepsilon}{2\|f\|+1}.$$

Define now $s := s_0 \vee 0$ and $t := t_0 \wedge \|f\|$. Then s, t are step functions and $0, s_0 \leq s \leq f \leq t \leq t_0, \|f\|$.

We get immediately that $t - s \leq t_0 - s_0$, so $\int (t-s) \leq \int (t_0 - s_0)$.

Moreover, since $s, f, t \geq 0$, it follows that $s^2 \leq f^2 \leq t^2$ and

$$\int (t^2 - s^2) = \int (t-s)(t+s) \leq \int 2 \cdot \|f\| (t-s) = 2\|f\| \cdot \int (t-s) \leq$$

$$\leq 2\|f\| \cdot \int (t_0 - s_0) < 2\|f\| \cdot \frac{\varepsilon}{2\|f\|+1} < \varepsilon.$$

Therefore, condition (ii) of the Riemann Criterion is satisfied and thus f^2 is integrable.

Now let f be arbitrary. Since $f^+, f^- \geq 0$, we get that $(f^+)^2, (f^-)^2$ are integrable. In view of $f^+ \cdot f^- = 0$, it follows that

$$f^2 = (f^+ - f^-)^2 = (f^+)^2 + (f^-)^2.$$

Apply now (ii) to conclude that f^2 is integrable.

(x) Apply (ii), (iii), (ix) and the fact that

$$fg = \frac{1}{4} \left((f+g)^2 - (f-g)^2 \right).$$

□

2. The Fundamental Theorem

Let $u, v \in [a,b]$, $u \leq v$ and let $c_{[u,v]} : [a,b] \to \mathbb{R}$ be the characteristic function of $[u,v]$:

$$c_{[u,v]}(x) = \begin{cases} 1 & \text{if } x \in [u,v] \\ 0 & \text{otherwise.} \end{cases}$$

We define similarly $c_{]u,v]}$.

Remark that $c_{[u,v]}$, $c_{]u,v]}$ are step functions and, therefore, they are integrable. It is easy to see that $\int_a^b c_{[u,v]} = \int_a^b c_{]u,v]}$.

If $f : [a,b] \to \mathbb{R}$ is integrable, then $c_{[u,v]} f$ is integrable by Prop 5.14-5.15.(x).

Now we define

$$\int_u^v f \overset{\text{def}}{=\!=} \int_a^b c_{[u,v]} f.$$

<u>Remark</u>

$$\int_a^b c_{[u,v]} f = \int_a^b c_{]u,v]} f.$$

<u>Proof</u>

Exercise. $\square$

<u>Proposition 2</u>

Let $f : [a,b] \to \mathbb{R}$ be integrable. Then for any $m \in [a,b]$,

$$\int_a^b f = \int_a^m f + \int_m^b f.$$

Proof.

Let $m \in [a,b]$. We remark first that $c_{[a,m]} + c_{]m,b]} = 1$. Then

$$\int_a^b f = \int_a^b 1 \cdot f = \int_a^b (c_{[a,m]} + c_{]m,b]}) f = \int_a^b c_{[a,m]} f + \int_a^b c_{]m,b]} f$$

$$= \int_a^m f + \int_m^b f.$$

∎

Remark

By taking $m = a$ in the above proposition, we get that $\int_a^a f = 0$.

Recall (Analysis I, Tutorial 15)

Definition

Let (X,d), (Y,ρ) be metric spaces and $f : X \to Y$. f is called uniformly continuous if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x,y \in X)\left(d(x,y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon \right).$$

Theorem

Let (X,d), (Y,ρ) be metric spaces and $f : X \to Y$. Then

f continuous and X compact $\Rightarrow$ f uniformly continuous.

Fundamental Theorem of Differential and Integral Calculus (compare with Theorem 5.18)

Let $f: [a,b] \to \mathbb{R}$ be a continuous function.

(i) f is integrable.

(ii) The function

$$F_0 : [a,b] \to \mathbb{R}, \quad F_0(x) \overset{\text{def}}{=} \int_a^x f$$

is an antiderivative of f, that is F_0 is differentiable and $F_0' = f$.

(iii) For any antiderivative F of f, the following holds

$$\int_a^b f = F(b) - F(a).$$

Proof

(i) Since f is continuous on the compact interval $[a,b]$, f is uniformly continuous.

Therefore for every $\varepsilon > 0$ there exists $\delta > 0$ s.t. for all $x, y \in [a,b]$

(1) $$|x-y| < \delta \Longrightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}.$$

Let $P = (a = x_0 < \dots < x_n = b)$ be any partition of $[a,b]$ s.t.
$\|P\| \overset{\text{def}}{=} \max\{x_i - x_{i-1} : i = 1, \dots, n\} < \delta$.

Then for any $i = 1, \dots, n$,

$$x, y \in [x_{i-1}, x_i] \Rightarrow |x - y| \le x_i - x_{i-1} \le \|P\| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a} \Rightarrow$$

(2) $$\Rightarrow f(x) - f(y) < \frac{\varepsilon}{b-a}.$$

Applying the Theorem of the Minimum and Maximum to the continuous function f on the compact interval $[x_{i-1}, x_i]$, $i = 1, \dots, n$, we get that there exist $m_i = \min f([x_{i-1}, x_i])$, $M_i = \max f([x_{i-1}, x_i])$ and, moreover, there are $u_i, v_i \in [x_{i-1}, x_i]$ with $f(u_i) = m_i$, $f(v_i) = M_i$.

Define now $s, t : [a,b] \to \mathbb{R}$ by

$$s(x) = m_i, \quad t(x) = M_i \quad \text{for } x \in [x_{i-1}, x_i[, \ i = 1, \dots, n$$

$$s(b) = t(b) = f(b).$$

Then s, t are step functions, $s \le f \le t$ and

$$\int t - \int s = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) - \sum_{i=1}^{n} m_i (x_i - x_{i-1}) =$$

$$= \sum_{i=1}^{n} (f(v_i) - f(u_i))(x_{i-1} - x_i) \overset{(2)}{<} \sum_{i=1}^{n} \frac{\varepsilon}{b-a} (x_i - x_{i-1})$$

$$= \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon.$$

Hence, we can apply the Riemann Criterion to conclude that f is integrable.

(ii) Let $x_0 \in [a,b]$. We have to show that $F_0'(x_0) = f(x_0)$, that is

(3) $$\lim_{\substack{x \to x_0 \\ x \ne x_0}} \frac{F_0(x) - F_0(x_0)}{x - x_0} = f(x_0).$$

By Proposition 4.5, (3) is equivalent with proving that for all $\varepsilon > 0$ there exists $\delta > 0$ s.t. for all $x \in [a,b]$

(4) $$0 < |x - x_0| < \delta \implies \left| \frac{F_0(x) - F_0(x_0)}{x - x_0} - f(x_0) \right| < \varepsilon.$$

Let $\varepsilon > 0$. Since f is continuous at x_0, there is $\delta > 0$ s.t. for all $x \in [a,b]$,

(5) $$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \frac{\varepsilon}{2}.$$

In the sequel we prove that (4) is satisfied with this δ.

Let $x \in [a,b]$ be such that $0 < |x - x_0| < \delta$. There are two cases:

1) $x < x_0$. Then

$$\left| \frac{F_0(x) - F_0(x_0)}{x - x_0} - f(x_0) \right| = \left| \frac{\int_a^x f - \int_a^{x_0} f}{x - x_0} - f(x_0) \right| =$$

$$= \left| \frac{\int_a^{x_0} f - \int_a^x f}{x_0 - x} - f(x_0) \right| \overset{\text{Prop. 2}}{=} \left| \frac{\int_x^{x_0} f - f(x_0)(x_0 - x)}{x_0 - x} \right| =$$

$$= \left| \frac{\int_x^{x_0} f - \int_x^{x_0} f(x_0)}{x_0 - x} \right| = \frac{\left| \int_x^{x_0} (f - f(x_0)) \right|}{x_0 - x} \le \frac{\int_x^{x_0} |f - f(x_0)|}{x_0 - x} \overset{(5)}{\le}$$

$$\le \frac{\frac{\varepsilon}{2}(x_0 - x)}{x_0 - x} = \frac{\varepsilon}{2} < \varepsilon.$$

2) $x > x_0$. We prove similarly that

$$\left| \frac{F_0(x) - F_0(x_0)}{x - x_0} - f(x_0) \right| = \left| \frac{\int_{x_0}^x f}{x - x_0} - f(x_0) \right| < \varepsilon.$$

Thus, F_0 is differentiable at x_0 and $F_0'(x_0) = f(x_0)$. Since $x_0 \in [a,b]$ was arbitrary, it follows that F_0 is differentiable and $F_0' = f$.

(iii) Let F be an antiderivative of f. Applying Proposition 4.41, it follows that there exists $c \in \mathbb{R}$ s.t. $F = F_0 + c$. Then

$$F(b) - F(a) = (F_0(b) + c) - (F_0(a) + c) = F_0(b) - F_0(a) =$$

$$= \int_a^b f - \int_a^a f = \int_a^b f.$$

$\square$

<u>Notation</u>

In place of $F(b) - F(a)$ one also writes $F\Big|_a^b$ or $F(x)\Big|_a^b$.

Proposition 5.13

Let $f: U_\rho(0) \to \mathbb{R}$ be a function given by a power series

$$f(x) = a_0 + a_1 x + \dots + a_n x^n + \dots = \sum_{n=0}^{\infty} a_n x^n,$$

which has the radius of convergence ρ.

Then the power series $\sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n$ obtained by term-wise passage to antiderivatives converges absolutely on $U_\rho(0)$ and defines a function $F: U_\rho(0) \to \mathbb{R}$ such that $F' = f$.

In particular, for $a < b$ in $U_\rho(0)$ we have $\int_a^b f = F(b) - F(a)$.

Proof

As in the proof of Theorem 4.11, by Remark 3.24(iii) we get that the power series $\sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n = \sum_{n=0}^{\infty} \frac{a_n}{n+1} \cdot x^{n+1}$ has the same radius of convergence ρ as $\sum_{n=0}^{\infty} a_n x^n$.

Hence $\sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n$ converges absolutely on $U_\rho(0)$ and thus defines a function $F: U_\rho(0) \to \mathbb{R}$, $F(x) = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n = \sum_{n=0}^{\infty} \frac{a_n}{n+1} \cdot x^{n+1}$.

Since $\left(\frac{a_n}{n+1} \cdot x^{n+1} \right)' = a_n x^n$ for all $n \geq 0$, applying Theorem 4.11 on the differentiability of power series, we get that $F' = f$.

The last claim follows immediately from the Fundamental Theorem (iii).

□