4 Type-2 Theory of Effectivity

Definition 4.1. *a) A (possibly partial)* multifunction $f :\subseteq X \rightrightarrows Y$ *is a subset of* $X \times Y$.
$\mathrm{dom}(f) := \{x \in X \mid \exists y \in Y : (x,y) \in f\}$ *and* $f(x) := \{y \in Y \mid (x,y) \in f\}$.

b) A Type-2 Machine *has an infinite read-only input tape, an infinite one-way output tape, and an unbounded work tape. It computes a (possibly partial) function* $F :\subseteq \{0,1\}^\omega \to \{0,1\}^\omega$.

c) A representation *of a set X is a partial surjective mapping* $\alpha :\subseteq \{0,1\}^\omega \to X$. *We call* $\bar{\sigma} \in \alpha$ *an* α–name *of* $\alpha(\bar{\sigma})$. *A point* $x \in X$ *is* α–computable *if it has a decidable α–name.*

d) Fix representations α of X and β of Y and a (possibly partial and multivalued) function $f :\subseteq X \rightrightarrows Y$. *A* (α,β)–realizer *of f is a (partial but single-valued) function* $F :\subseteq \{0,1\}^\omega \to \{0,1\}^\omega$ *with* $f\big(\alpha(\bar{\sigma})\big) \ni \beta\big(F(\bar{\sigma})\big)$ *for every* $\bar{\sigma} \in \mathrm{dom}(F) := \{\bar{\sigma} \mid \alpha(\bar{\sigma}) \in \mathrm{dom}(f)\}$.

e) A function as in d) is (α,β)–computable *if it has a computable* (α,β)*–realizer.*
It is (α,β)–continuous *if it has a continuous realizer.*

f) We say that $U \subseteq X$ *is* α–r.e. *if there exists a Turing machine which terminates precisely on input of all α–names of* $\vec{x} \in U$ *and diverges on all α–names of* $\vec{x} \in X \setminus U$.

Example 4.2 *a) Define a* ρ*–name*[⋆] *of* $x \in \mathbb{R}$ *to be a sequence* $a_n \in \mathbb{Z}$ *(encoded in binary) such that* $|x - a_n/2^{n+1}| \leq 2^{-n}$.
b) Define a ρ_C*–name of* $x \in \mathbb{R}$ *to be two sequences* $q_n, \varepsilon_n \in \mathbb{Q}$ *such that* $|x - q_n| < \varepsilon_n \to 0$.
c) A $\rho_<$*–name of* $x \in \mathbb{R}$ *is a sequence* $b_n \in \mathbb{Z}$ *with* $\sup_n b_n/2^{n+1} = x$;
a $\rho_>$*–name of* $x \in \mathbb{R}$ *is a sequence* $c_n \in \mathbb{Z}$ *with* $\inf_n c_n/2^{n+1} = x$.
d) Define a ρ_n*–name of* $x \in \mathbb{R}$ *to be a sequence* $a_n \in \mathbb{Z}$ *such that* $\lim_n a_n/2^{n+1} = x$.
e) Define a ν*–name of* $y \in \mathbb{N}$ *to be the string* $\mathtt{1}^y\mathtt{0}^\omega$. *Define a* ν_b*–name of* $y \in \mathbb{N}$ *to be the string* $(b_0, \mathtt{0}, b_1, \mathtt{0}, \ldots, b_{n-1}, \mathtt{0}, \mathtt{1}^\omega)$ *where* $y = b_0 + 2b_1 + \cdots + 2^{n-1}b_{n-1} + 2^n - 1$.

Theorem 4.3. *a) Every (oracle-)computable* $F :\subseteq \{0,1\}^\omega \to \{0,1\}^\omega$ *is continuous.*
b) To every continuous $F :\subseteq \{0,1\}^\omega \to \{0,1\}^\omega$,
there exists an oracle relative to which F becomes computable.
c) Every oracle-computable $f : [0;1] \to \mathbb{R}$ *is continuous!*
d) To every continuous $f : \mathbb{R} \to \mathbb{R}$ *there is an oracle relative to which f becomes computable.*
e) Every (relatively) ρ*–r.e. set* $U \subseteq \mathbb{R}$ *is open.*
f) Every open $U \subseteq \mathbb{R}$ *is relatively* ρ*–r.e.*
g) The identity $\mathrm{id} : \mathbb{N} \to \mathbb{N}$ *is both* (ν, ν_b)*–computable and* (ν_b, ν)*–computable.*

4.1 Constructing with, and Comparing, Representations

Definition 4.4. *a) Write* $\alpha \preceq \beta$ *if* $\mathrm{id} : X \to X$ *is* (α,β)*–computable.*
b) Let α_i *be representations for* X_i, $i \in I \subseteq \mathbb{N}$, *and* $\langle \cdot \mid \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *a computable surjective pairing function. Define* $(\sigma_m)_m$ *to be a* $\big(\prod_{i \in I} \alpha_i\big)$*–name of* $(x_i)_i \in \prod_i X_i$ *iff* $(\sigma_{\langle i,n \rangle})_n$ *is an* α_i*–name of* $x_i \in X_i$ *for every* $i \in I$.
c) For representations α,β *of X let* $\alpha \sqcap \beta := (\alpha \times \beta)\big|^{\Delta_X}$, *where* $\Delta_X := \{(x,x) \mid x \in X\}$.

⋆ This is subtly different from the representation denoted by ρ in [7] *d) A* name *of a continuous partial* $F :\subseteq \{0,1\}^\omega \to \{0,1\}^\omega$ *is a monotone* $\hat{f} : \{0,1\}^* \to \{0,1\}^*$ *(enumerating its table of arguments and values as some* $f \in \{0,1\}^\omega$*) with* $f_\omega\big|_{\mathrm{dom}(F)} = F$.

e) Fix representations α *of* X *and* β *of* Y*. The representation* $[\alpha\to\beta]$ *of the set* $R_{\alpha,\beta}[X,Y]$ *of all* (α,β)*–realizable total* $g : X \to Y$ *is defined as follows: A* $[\alpha\to\beta]$*–name of* g *is a name of an* (α,β)*–realizer of* g*.*

f) For multifunctions $f :\subseteq X \rightrightarrows Y$ *and* $g :\subseteq Y \rightrightarrows Z$*, their* composition *is defined as*

$$g\circ f \ := \ \big\{(x,z)\,\big|\, x\in X, z\in Z, f(x)\subseteq \mathrm{dom}(g),\ \exists y\in Y : (x,y)\in f \wedge (y,z)\in g\big\} \ . \tag{1}$$

Proposition 4.5. *a) Let* α,β,γ *denote representations of* X,Y,Z*, respectively. If* $f :\subseteq X \rightrightarrows Y$ *is* (α,β)*–computable and* $g :\subseteq Y \rightrightarrows Z$ *is* (β,γ)*–computable, then their composition* $g\circ f$ *is* (α,γ)*–computable.*

b) For $\alpha\preceq\alpha'$ *and* $\beta\preceq\beta'$, $[\alpha'\to\beta]\preceq[\alpha\to\beta']$.

c) Let α *be a representation of* X*. Then* $\alpha^\omega\preceq[\nu\to\alpha]\preceq\alpha^\omega$.

d) Fix representations α *of* X *and* β *of* Y *and* γ *of* $R_{\alpha,\beta}[X,Y]$.
$R_{\alpha,\beta}[X,Y]\times X \ni (g,x)\mapsto g(x)\in Y$ *is* $(\gamma\times\alpha,\beta)$*–computable iff* $\gamma\preceq[\alpha\to\beta]$.

e) Fix representations α *of* X *and* β *of* Y *and* γ *of* Z*. Then type conversion*

$$R_{\alpha\times\beta,\gamma}[X\times Y,Z]\times X \ \ni \ (g,x) \ \mapsto \ \big(Y\ni y\mapsto g(x,y)\in Z\big) \ \in \ R_{\beta,\gamma}[Y,Z] \tag{2}$$

is well-defined and $\big([\alpha\times\beta\to\gamma]\times\alpha,[\beta\to\gamma]\big)$*–computable.*

f) Also the converse conversion

$$\begin{aligned} R_{\alpha,[\beta\to\gamma]}\big[X,R_{\beta,\gamma}[Y,Z]\big] \ &\ni \ \big(X\ni x\mapsto g(x,\cdot)\in R_{\beta,\gamma}[Y,Z]\big) \\ &\mapsto \ \big(X\times Y\ni(x,y)\mapsto g(x,y)\in Z\big) \ \in \ R_{\alpha\times\beta,\gamma}[X\times Y,Z] \end{aligned}$$

is well-defined and $\big(\big[\alpha\to[\beta\to\gamma]\big],[\alpha\times\beta\to\gamma]\big)$*–computable.*

4.2 Representing real functions and closed subsets

Definition 4.6. *a) The representation* $\widehat{[\rho^d\to\rho]}$ *of* $f\in C(\mathbb{R}^d)$ *is defined as follows: A name is a double sequence* $P_{n,m}\in\mathbb{D}[X_1,\ldots,X_d]$ *with* $|f(\vec{x})-P_{n,m}(\vec{x})|\le 2^{-n}$ *for all* $\|\vec{x}\|\le m$*.*

b) A nonempty closed set $A\subseteq\mathbb{R}^d$ *is* computable *if the function*

$$\mathrm{dist}_A : \mathbb{R}^d \ \ni \ \vec{x} \ \mapsto \ \min\big\{\|\vec{x}-\vec{a}\| : \vec{a}\in A\big\} \ \in \ \mathbb{R} \tag{3}$$

is computable. A ψ^d*–name of* $A\in\mathcal{A}^{(d)}$ *is a* $[\rho^d\to\rho]$*–name of* dist_A*,*
where $\mathcal{A}^{(d)}$ *denotes the space of nonempty closed subsets of* $\mathbb{R}^d$*.*

c) A $\psi^d_<$*–name of A is a* $\big(\prod_{m\in\mathbb{N}}\rho^d\big)$*–name of some sequence* $\vec{x}_m\in A$ *dense in A.*

d) A $\psi^d_>$*–name of A are two sequences* $\vec{q}_n\in\mathbb{Q}^d$ *and* $\varepsilon_n\in\mathbb{Q}$ *such that*

$$\mathbb{R}^d\setminus A \ = \ \bigcup\nolimits_n B(\vec{q}_n,\varepsilon_n) \quad \textit{where} \quad B(\vec{x},r):=\{\vec{y}:\|\vec{x}-\vec{y}\|<r\} \ . \tag{4}$$

Theorem 4.7. *a) It holds* $\rho \preceq \rho_< \sqcap \rho_> \preceq \rho_C \preceq \rho$ *and* $[\rho^d \to \rho] \preceq [\rho^d \to \rho_<] \sqcap [\rho^d \to \rho_>] \preceq [\rho^d \to \rho]$.

b) It holds $R_{\rho^d,\rho}[\mathbb{R}^d,\mathbb{R}] = C(\mathbb{R}^d,\mathbb{R})$ *and* $[\widehat{\rho^d \to \rho}] \preceq [\rho^d \to \rho] \preceq [\widehat{\rho^d \to \rho}]$

c) Every $(\rho,\rho_<)$*–computable* $f:[0;1]\to\mathbb{R}$ *is lower semi-continuous.*

d) A set $A \in \mathcal{A}^{(d)}$ *is* $\psi^d_>$*–computable iff* $\mathbb{R}^d \setminus A$ *is* ρ^d*–r.e.*

e) Let $\|\cdot\|$ *in Equation (4) denote any fixed computable norm.*
Let $\|\cdot\|'$ *denote some other norm on* $\mathbb{R}^d$ *with induced representation* $\psi'^d_>$.
Then i) $\psi^d_> \preceq \psi'^d_>$ *and ii)* $\psi^d_< \preceq \psi'^d_<$.

f) It holds $\psi^d \preceq \psi^d_< \sqcap \psi^d_> \preceq \psi^d$.
Moreover A is $\psi^d_<$*–computable iff* dist_A *is* $(\rho^d,\rho_>)$*–computable;*
and A is $\psi^d_>$*–computable iff* dist_A *is* $(\rho^d,\rho_<)$*–computable.*

g) Union $\mathcal{A}^{(d)} \times \mathcal{A}^{(d)} \ni (A,B) \mapsto A\cup B \in \mathcal{A}^{(d)}$ *is* $(\psi^d\times\psi^d,\psi^d)$*–computable;*
but intersection is not.

h) Closed image $C(\mathbb{R}^d,\mathbb{R}^k)\times\mathcal{A}^{(d)} \ni (f,A) \mapsto \overline{f[A]} \in \mathcal{A}^{(k)}$ *is* $([\rho^d\to\rho^k]\times\psi^d_<,\psi^k_<)$*–computable.*

j) Preimage $C(\mathbb{R}^d,\mathbb{R}^k)\times\mathcal{A}^{(k)} \ni (f,B) \mapsto f^{-1}[B] \in \mathcal{A}^{(d)}$ *is* $([\rho^d\to\rho^k]\times\psi^k_>,\psi^d_>)$*–computable.*

k) $\{A\in\mathcal{A}^{(d)} : A\cap[0,1]^d=\emptyset\}$ *is* $\psi^d_>$*–r.e.*

6 Real Complexity Theory

6.2 Parameterized Type-2 Function Complexity

Definition 6.1. *a) A partial function* $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ *is* computable in time $t:\mathbb{N}\to\mathbb{N}$ *if a Type-2 Machine can, given* $\bar\sigma \in \mathrm{dom}(F)$*, produce* $\bar\tau = F(\bar\sigma)$ *sucht that the n-th symbol of* $\bar\tau$ *appears within* $t(n)$ *steps.*

b) For spaces X and Y with representations α *and* β*, a partial multivalued* $f:\subseteq X \rightrightarrows Y$ *is computable* in time $t(n)$ *if it admits an* (α,β)*–realizer* $F:\subseteq\Sigma^\omega\to\Sigma^\omega$ *computable in time* $t(n)$.

Example 6.2 *a) If* $F:\subseteq\Sigma^\omega\to\Sigma^\omega$ *is computable in time t, it has t as modulus of continuity.*

b) If $f:[0;1]\to[0;1]$ *is* (ρ_C,ρ)*–computable in time t for some* $t:\mathbb{N}\to\mathbb{N}$, *f is constant.*

c) Every computable $f:[0;1]\to\mathbb{R}$ *is* (ρ,ρ_C)*–computable in quadratic time.*

d) The following C^∞ *'pulse' function is computable in polynomial time:*

$$[-1;1]\ni x\mapsto \exp\left(-\tfrac{x^2}{1-x^2}\right),\quad [-1;1]\not\ni x\mapsto 0$$

e) If $F:\subseteq\Sigma^\omega\to\Sigma^\omega$ *is computable and* $\mathrm{dom}(F)$ *compact, then F is computable in some recursive time bound* $t:\mathbb{N}\to\mathbb{N}$ *depending on the output precision n only.*

f) Inversion $[2^{-K};1]\ni x\mapsto 1/x$ *is* (ρ,ρ)*–computable in time polynomial in* $n+K$.

6.3 Second-Order Representations

For $L>0$ and for metric spaces (X,d) and (Y,e) let

$$\mathrm{Lip}_L(X,Y) := \left\{f:X\to Y : e\big(f(x),f(x')\big)\le L\cdot d(x,y)\right\},\quad \mathrm{Lip}(X,Y) := \bigcup_{L>0}\mathrm{Lip}_L(X,Y) \ .$$

We may omit Y in case $Y=\mathbb{R}$.

Problem 6.3 *a) According to Example 6.2a), the evaluation operator* $(F,\bar{\sigma}) \mapsto F(\bar{\sigma})$ *admits no upper running time bound depending on the output precision n only.*

b) Moreover, even restricted to (the compact set of) non-expansive $F : \{\mathtt{0},\mathtt{1}\}^\omega \to \{\mathtt{0},\mathtt{1}\}^\omega$, *the encoding of* $F = f_\omega$ *via the table of values of* $f : \{\mathtt{0},\mathtt{1}\}^n \to \{\mathtt{0},\mathtt{1}\}^n$ *makes evaluation computable in exponential time but no better.*

c) In fact the class $\mathrm{Lip}_1\big([0;1],[0;1]\big)$ *of 1-Lipschitz* $f : [0;1] \to [0;1]$ *does not admit a representation rendering evaluation* $(f,x) \mapsto f(x)$ *computable in subexponential time.*

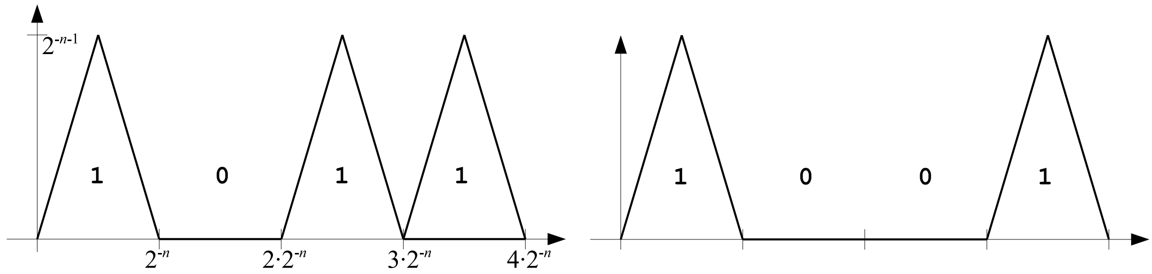

Fig. 1. Encoding the binary strings `1011` and `1001` into 1-Lipschitz functions f,g with $\|f-g\|$ not too small

Note that an implementation of the evaluation operator (e.g. in `iRRAM`) would *not* reasonably be provided with a function argument f as an infinite binary string but via 'oracle' access to approximate dyadic evaluation queries

$$\mathbb{Z}\times\mathbb{N} \ni (a,2^n) \Rightarrow b \in \mathbb{N} \text{ s.t. } \big|f(a/2^{n+1}) - b/2^{n+1}\big| \le 1/2^n \quad .$$

Definition 6.4. *a) An* oracle Type-2 Machine $\mathcal{M}^\Psi$ *may write onto its query tape some* $\vec{w} \in \Sigma^*$ *which, when entered the designated query state, will be replaced with* $\vec{v} := \psi(\vec{w})$.
(We implicitly employ some linear-time bicomputable self-delimited encoding on this tape such as $(w_1,\ldots,w_n) \mapsto \mathtt{1}\,w_1\,\mathtt{1}\,w_2\ldots\mathtt{1}\,w_n\,\mathtt{0}$.*)*

b) $\mathcal{M}^?$ computes *a partial mapping* $\tilde{F} :\subseteq (\Sigma^*)^{\Sigma^*} \to (\Sigma^*)^{\Sigma^*}$ *if, for every* $\psi \in \mathrm{dom}(\tilde{F})$, $\mathcal{M}^\Psi$ *on input* $\vec{v} \in \Sigma^*$ *produces* $\tilde{F}\big(\psi\big)(\vec{v}) \in \Sigma^*$ *and terminates.*

c) Let $\mathrm{LM} \subsetneq (\Sigma^*)^{\Sigma^*}$ *denote the set of all total functions* $\psi : \Sigma^* \to \Sigma^*$ *length-monotone in the sense of verifying*

$$|\vec{v}| \le |\vec{w}| \quad \Rightarrow \quad |\psi(\vec{v})| \le |\psi(\vec{w})| \quad . \tag{5}$$

Write $|\psi| : \mathbb{N} \to \mathbb{N}$ *for the (thus well-defined) mapping* $|\vec{w}| \mapsto |\psi(\vec{w})|$.

d) A second-order representation *for a space X is a surjective partial mapping* $\tilde{\xi} :\subseteq \mathrm{LM} \to X$.

Example 6.5 *a) Any ordinary representation* $\xi :\subseteq \{\mathtt{0},\mathtt{1}\}^\omega \to X$ induces *a second-order representation* $\tilde{\xi}$ *as follows: Whenever* $\bar{\sigma}$ *is a* ξ*–name of x, then* $\psi : \Sigma^* \ni \vec{v} \mapsto \sigma_{|\vec{v}|} \in \Sigma$ *is a* $\tilde{\xi}$*–name of said x.*

b) (Re-)define a $\tilde{\rho}$*–name of* $x \in \mathbb{R}$ *to be a length-monotone mapping* $\psi : \{\mathtt{0},\mathtt{1}\}^* \to \{\mathtt{0},\mathtt{1}\}^*$ *s.t.* $\left|x - \frac{\mathrm{bin}\big(\psi(\vec{w})\big)}{2^{|\vec{w}|+1}}\right| \le 2^{-|\vec{w}|}$ *for all* $\vec{w}$.

c) *Define a second-order representation* $\rho^{\mathbb{D}}$ *of* $C[0;1]$ *as follows:* $\psi \in \mathrm{LM}$ *is a* $\rho^{\mathbb{D}}$*-name of* $f \in C[0;1]$ *if, for all* $\vec{w} \in \Sigma^*$*, it holds*

$$\left|\frac{\mathrm{bin}\big(\psi(\vec{w})\big)}{2^{|\vec{w}|+1}} - f\left(\frac{\mathrm{bin}(\vec{w})}{2^{|\vec{w}|+1}}\right)\right| \leq 2^{-|\vec{w}|} \quad . \tag{6}$$

d) *Define a second-order representation* $\rho^{\mathbb{D}} \sqcap L$ *of* $\mathrm{Lip}[0;1]$ *by saying that, whenever* ψ *is a* $\rho^{\mathbb{D}}$*–name of* $f \in \mathrm{Lip}_{2^\ell}[0;1]$*, then* $\zeta : \Sigma^* \ni \vec{w} \mapsto \mathtt{1}^\ell\,\mathtt{0} \circ \psi(\vec{w}) \in \Sigma^*$ *is a* $\rho^{\mathbb{D}} \sqcap L$*–name of* f*.*

e) *Define a* $\widetilde{[\rho{\to}\rho]}$*–name*** ψ *of* $f \in C[0;1]$ *to be a mapping* $\Sigma^* \ni \vec{w} \mapsto \mathtt{1}^{\mu(|\vec{w}|)}\,\mathtt{0}\,\psi(\vec{w}) \in \Sigma^*$*, where* ψ *denotes a* $\rho^{\mathbb{D}}$*–name of* f *and* $\mu : \mathbb{N} \to \mathbb{N}$ *is a modulus of uniform continuity to it.*

How 'long' are these names asymptotically? Relate $|\psi|(n)$ to quantitative properties of f. What is $|\psi|$ for names ψ w.r.t. a second-order representation induced by a first-order one?

6.4 Second-Order Polynomial-Time Complexity

Note that an oracle query $\vec{w} \mapsto \vec{v} := \psi(\vec{w})$ according to Definition 6.4b) may return a (much) longer answer for one argument ψ than for another ψ'. So in order to be able to even read such a reply, we have to consider as 'polynomial' a running time bound that depends on both n and $|\psi|$. The former being an integer and the latter an integer function, suggests

Definition 6.6. *a) A* second-order *polynomial* $P = P(n, \lambda)$ *is a term composed from variable symbol* n*, unary function symbol* $\lambda()$*, binary function symbols* $+$ *and* $\times$*, and positive integer constants.*

b) *Let* $T : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ *be arbitrary. Oracle machine* $\mathcal{M}^?$ *computing* $\tilde{F} :\subseteq \mathrm{LM} \to \mathrm{LM}$ *according to Definition 6.4 operates in* time T *if, for every* $\psi \in \mathrm{dom}(\tilde{F})$ *and every* $\vec{v} \in \Sigma^*$*,* $\mathcal{M}^\psi$ *on input* $\vec{v}$ *produces* $F\big(\psi\big)(\vec{v})$ *and terminates within at most* $T(|\vec{v}|, |\psi|)$ *steps.*

c) *For second-order representations* $\tilde{\tilde{\xi}}$ *of* X *and* $\tilde{\upsilon}$ *of* Y*, a (possibly partial and multivalued) function* $f :\subseteq X \rightrightarrows Y$ *is* $(\tilde{\tilde{\xi}}, \tilde{\upsilon})$*–computable in time* T *iff* f *has a* $(\tilde{\tilde{\xi}}, \tilde{\upsilon})$*–realizer* $\tilde{F}$ *computable in this time.*

d) Second-order polytime *computability means computability in time* P *for some second-order polynomial* P*.*

Computations on 'long' names ψ are thus allotted more time and still considered polynomial.

Example 6.7 *a)* $\lambda\big(n + \lambda(n^2 + 3n) \cdot n \cdot \lambda^3(n)\big) \cdot n$ *is a second-order polynomial.*

b) *Second-order polynomials are closed under* both *kinds of composition:*

$$\big(Q \circ P\big)(n, \lambda) \;:=\; Q\big(P(n,\lambda), \lambda\big) \quad \textit{and} \quad \big(Q \bullet P\big)(n, \lambda) \;:=\; Q\big(n, P(\cdot, \lambda)\big) \quad . \tag{7}$$

c) *Addition and multiplication are* 2^{nd}*-order polytime* $\tilde{\rho}$*–computable on* $\mathbb{R}$*; but* $x \mapsto e^x$ *is not.*

d) *Fix ordinary representations* ξ *of* X *and* υ *of* Y *with induced second-order representations* $\tilde{\tilde{\xi}}$ *and* $\tilde{\upsilon}$*. Then* $f :\subseteq X \rightrightarrows Y$ *is polytime* (ξ, υ)*–computable iff it is second-order polytime* $(\tilde{\tilde{\xi}}, \tilde{\upsilon})$*–computable.*

** in [3] called a $\delta_\square$–name

e) *Evaluation* $\mathrm{Lip}[0;1]\times[0;1]\ni(f,x)\mapsto f(x)$ *is* $(\rho^{\mathbb{D}}\sqcap L\times\tilde{\rho},\tilde{\rho})$*–computable in second-order polytime.*
f) *Evaluation* $C[0;1]\times[0;1]\ni(f,x)\mapsto f(x)$ *is* $(\widetilde{[\rho\to\rho]}\times\tilde{\rho},\tilde{\rho})$*–computable in second-order polytime.*

Definition 6.8. *a) Write* PRED $\subseteq$ LM *for the class of* $\psi:\Sigma^*\to\{0,1\}$.
b) *Let* $\mathcal{P}^2$ *denote the class of* $\tilde{F}:\subseteq \mathrm{LM}\to\mathrm{PRED}$ *computable by an oracle Type-2 Machine in second-order polytime.*
c) *Let* $\mathcal{NP}^2$ *denote the class of* $\tilde{F}:\subseteq \mathrm{LM}\to\mathrm{PRED}$ *computable by a non-deterministic oracle Type-2 Machine in second-order polytime.*
d) *We may identify a* $\tilde{F}:\subseteq \mathrm{LM}\to\mathrm{PRED}$ *with the set* $\{(\psi,\vec{v}):\psi\in\mathrm{dom}(\tilde{F}),\vec{v}\in\Sigma^*,\tilde{F}(\psi,\vec{v})=1\}$ *considered as a* promise *(second-order decision) problem.*

Example 6.9 *The following problem* EXIST^2 *belongs to* $\mathcal{NP}^2$ *but not to* $\mathcal{P}^2$*:*

$$\big\{(P,\vec{x})\ :\ P\in\mathrm{PRED},\ \exists\vec{y}\in\Sigma^{|\vec{x}|}:\ P(\langle\vec{x},\vec{y}\rangle)=1\big\}$$

6.5 Reductions

Definition 6.10. *Fix spaces* A,B,X,Y *with respective (ordinary or second-order) representations* $\alpha,\beta,\xi,\upsilon$. *Consider (possibly multivalued but total) functions* $f:A\rightrightarrows B$ *and* $g:X\rightrightarrows Y$.

a) *Call* f computably $(\alpha,\beta,\xi,\upsilon)$–reducible *to* g *if there exist multi-functions* $r:A\rightrightarrows X$, (α,ξ)*–computable, and* $s:Y\times A\rightrightarrows B$, $(\upsilon\times\alpha,\beta)$*–computable, such that for all* $a\in A$ *it holds* $s\Big(g\big(r(a)\big),a\Big)\subseteq f(a)$.
b) *The above functions* r *and* s *constitute a* (second-order) polytime reduction *if they are (second-order) polytime computable.*

Example 6.11 *Consider the following multivalued mappings:*

– $\mathrm{LLPO}:\{0,1\}^\omega\ni\bar{\sigma}\rightrightarrows\begin{cases}\{0^\omega\} & :\ \exists!n:\sigma_n=1,\ n\text{ even}\\ \{1^\omega\} & :\ \exists!n:\sigma_n=1,\ n\text{ odd}\\ \{0^\omega,1^\omega\} & :\ \forall n:\sigma_n=0\\ \{\} & :\ \exists n\neq m:\sigma_n=1=\sigma_m\end{cases}$
– $\mathsf{B}_\mathrm{I}:\{(a,b):0\le a\le b\le 1\}\ni(a,b)\rightrightarrows y\in[a,b]\subseteq[0;1]$
– $\mathsf{IVT}:\big\{f:[0;1]\to[-1;1]\text{ continuous s.t. }f(0)<0<f(1)\big\}\ni f\rightrightarrows x\in f^{-1}[0]\subseteq[0;1]$

a) *Every* $\tilde{F}\in\mathcal{NP}^2$ *is second-order polytime reducible to* EXIST^2.
b) $\mathrm{MAX}:C[0;1]\to C[0;1]$ *is 2nd-order polytime* $\big(\widetilde{[\rho\to\rho]},\widetilde{[\rho\to\rho]},\mathrm{id},\mathrm{id}\big)$*–reducible to* EXIST^2
c) *and* EXIST^2 *is second-order polytime* $\big(\mathrm{id},\mathrm{id},\widetilde{[\rho\to\rho]},\widetilde{[\rho\to\rho]}\big)$*–reducible to* $\mathrm{MAX}|_{C^\infty[0;1]}$.
d) LLPO *is not computable;*
B_I *is not* $(\rho_<\times\rho_>,\rho)$*–computable and* IVT *is not* $\big([\rho\to\rho],\rho\big)$*–computable.*
e) LLPO *is* $(\mathrm{id},\mathrm{id},\rho_<\times\rho_>,\rho)$*–reducible to* B_I.
f) IVT *is* $\big([\rho\to\rho],\rho,\rho_<\times\rho_>,\rho\big)$*–reducible to* B_I.
g) B_I *is* $\big(\rho_<\times\rho_>,\rho,[\rho\to\rho],\rho\big)$*–reducible to* IVT.

7 A View on the Practical Side: `iRRAM`

- Restricted to $f : [0;1] \to \mathbb{R}$ with $f(0) < 0 < f(1)$ and a *unique* root, this root can be found computably: How?
- `REAL` semantics provided by `iRRAM` via automatic re-iteration
- multivalued intrinsic functions;
  e.g. `bool bound (const REAL& x, const long k)` where

$$\texttt{bound}(x,k) = \begin{cases} \texttt{true} & : |x| \leq 2^{k-2} \\ \texttt{false} & : |x| > 2^k \\ \texttt{true}\text{ or }\texttt{false} & : 2^k \geq |x| > 2^{k-2} \end{cases}$$

- lazy Booleans and branching on multivalued tests

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