Computable Urysohn

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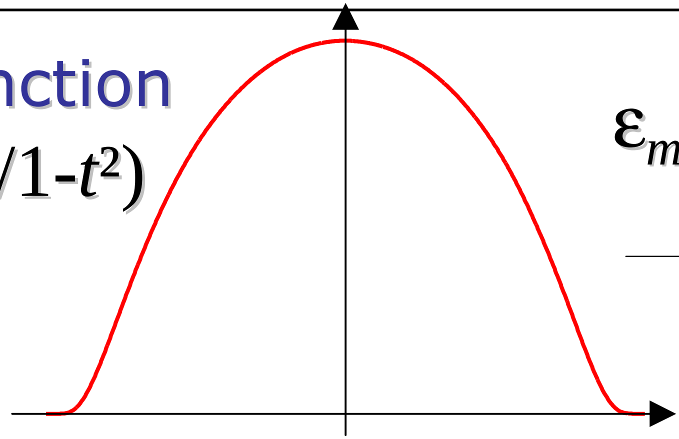
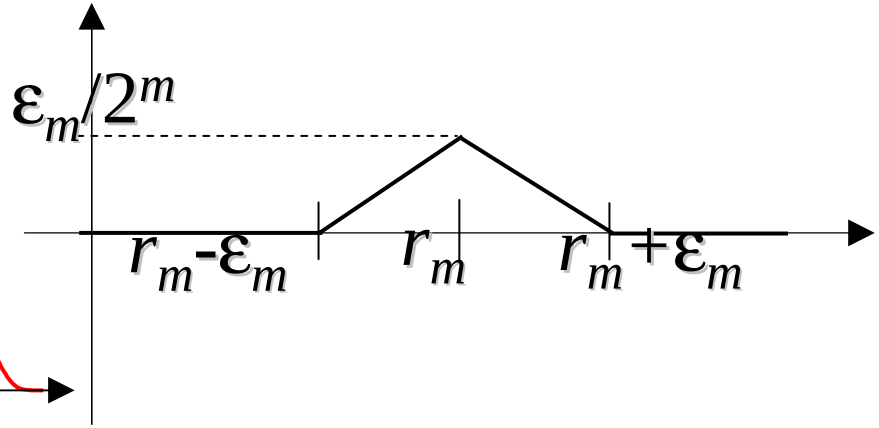
C^∞ 'pulse' function

$\varphi(t) = \exp(-t^2/1-t^2)$

$|t|<1$





Let $(r_m)_m, (\varepsilon_m)_m \subseteq \mathbb{Q}$ be computable sequences

Then there is a computable C^∞ $f:[0;1]\to[0;1]$

s.t. $f^{-1}[0] = [0;1]\setminus\bigcup_m (r_m-\varepsilon_m, r_m+\varepsilon_m)$.

Proof: Let $f(x):= \sum_m \max(0,\varepsilon_m-|x-r_m|)/2^m$

Specker'59: Uncomputable roots

approximating a root
vs. approximate root

Lemma: There are computable sequences $(r_m)_m, (\varepsilon_m)_m \subseteq \mathbb{Q}$ s.t. $U := \bigcup_m (r_m\text{-}\varepsilon_m, r_m+\varepsilon_m)$ contains all computable reals in [0;1] and has measure <½.

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Corollary: There is a computable C∞ $f:[0;1]\to[0;1]$ s.t. $f^{-1}[0]$ has measure >½ but contains no computable real number.

Singular Covering of Computable Reals

Lemma: There are computable sequences $(r_m)_m, (\varepsilon_m)_m \subseteq \mathbb{Q}$ s.t. $U := \bigcup_m (r_m-\varepsilon_m, r_m+\varepsilon_m)$ contains all computable reals in [0;1] and has measure $<\frac{1}{2}$.

Machine computes $r\in\mathbb{R}$ iff prints seq. $a_n\subseteq\mathbb{Z}$ with $|a_n/2^{n+1} - a_m/2^{m+1}| \leq 2^{-n} + 2^{-m}$.

Proof: Dove-tailing w.r.t. (M,t):
If Turing machine $\#M$ within t (but not t-1) steps prints $a_1,\ldots a_{M+5}$ s.t. $|a_k/2^{k+1}-a_\ell/2^{\ell+1}| \leq 2^{-k}+2^{-\ell}$ $\forall 1\leq k,\ell\leq M+5$
then let $r_{\langle M,t\rangle} := a_{M+5}/2^{M+6}$ and $\varepsilon_{\langle M,t\rangle} := 2^{-M-5}$,
else $r_{\langle M,t\rangle} := 0$ and $\varepsilon_{\langle M,t\rangle} := 2^{-\langle M,t\rangle-3}$.